Euclidean Special Relativity

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Abstract

A Euclidean interpretation of special relativity is given wherein proper time τ acts as the fourth Euclidean coordinate, and time t becomes a fifth Euclidean dimension. Velocity components in both space and time are formalized while their vector sum in four dimensions has invariant magnitude c. Classical equations are derived from this Euclidean concept.

1 Introduction

Euclidean relativity, both special and general, is steadily gaining attention as a viable alternative to the Minkowski framework, after the works of a number of authors. Amongst others Montanus [1,2], Gersten [3] and Almeida [4], have paved the way. Its history goes further back, as early as 1963 when Robert d'E Atkinson [5] first proposed Euclidean general relativity.

The version in the present paper emphasizes extending the notion of velocity to the time dimension. The consistency of this concept in 4D Euclidean space is shown with the classical Lorentz transformations. Following paragraphs show how the new approach can give meaningful explanations for various relativistic effects, after which some Euclidean 4-vectors are established.

A simplified and popularized version of this concept is also available on the web at http://www.euclideanrelativity.com.

2 The Time Dimension

Minkowski interpretations of special relativity treat time differently from spatial dimensions, showing from the Minkowski metric where the time component is given the opposite sign. Some alternative interpretations (*e.g.* [1-4]) seek positive definite Euclidean metrics for space-time.

If time is considered a fourth spatial dimension, then it must show properties similar to those found in the other three. In there we encounter properties like length, speed, acceleration, curvature *etc.*, expressed respectively as s, ds/dt, d^2s/dt^2, R^a_{bcd} *etc.* Of those properties, a single one can be measured relatively easily in the time dimension: the 'length' or timeduration Δt. That raises the question of how a hypothetical speed in time, let us call it χ, should be expressed mathematically. In [6], Greene has given a derivation of an expression that can be used as the velocity component in the Euclidean time dimension. Rewriting the usual Minkowski invariant

$$c^2 = (dct/d\tau)^2 - (dx/d\tau)^2 - (dy/d\tau)^2 - (dz/d\tau)^2 \tag{1}$$

into Euclidean form:

$$c^2 = (cd\tau/dt)^2 + (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 \tag{2}$$

one arrives at the temporal velocity component

$$\chi = cd\tau/dt \tag{3}$$

This clearly defines τ as the coordinate for the fourth Euclidean dimension, and it says that the velocity components in all *four* dimensions involve derivatives with respect to t, which then can no longer represent the fourth dimension. It can only be an extra, fifth dimension, x_5 (provided we index the other four x_1, x_2, x_3, and x_4 respectively, with $\tau = x_4$). This fifth dimension is sometimes treated as a parameter in Euclidean approaches similar to special relativity, *e.g.* in [1,2], but here it will be treated as a genuine extra Euclidean dimension. A general expression for speed in the time dimension (henceforth refereed to as time-speed) is now:

$$\chi = cdx_4/dx_5 \tag{4}$$

while the scalar value of time-speed χ is

$$\chi = \sqrt{c^2 - v^2} \tag{5}$$

The general expression for spatial velocity components in 4D Euclidean space-time is

$$v_i = dx_i/dx_5 \tag{6}$$

3 Using Time-Speed in Special Relativity

It will be shown that the Lorentz transformation equations for length and time can be reproduced from the Euclidean context.

Maintaining orthogonality for all Euclidean dimensions, Eqs. (2) and (5) imply that the axes for the proper time dimension and the spatial dimension in the direction of the initial motion must have *rotated* for the moving object, as seen from the rest frame of the observer, in fact defining Lorentz transformations as rotations in SO(4). See also [1], where this is referred to as a Relative Euclidean Space-Time. In the approach that follows now, these axes will therefor (unlike in the Minkowski diagram) both rotate in the same direction, clockwise or counter clockwise, depending on the direction of the motion. The diagrams in Fig. 1 and Fig. 2 should illustrate this.

Figure 1 depicts an object A at rest together with an observer at O, also at rest. The horizontal axis

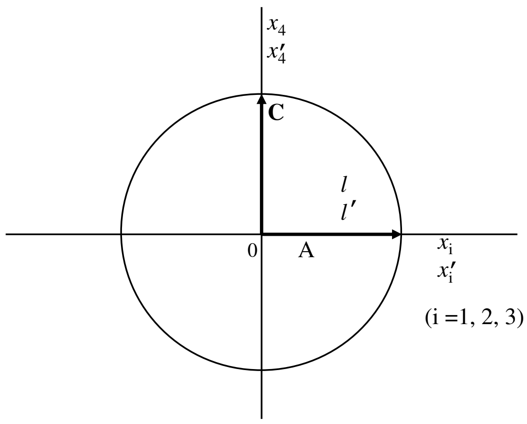

Figure 1: 4D representation of an observer at O and an object A, both at rest.

shows both the spatial dimensions x'_i, $i = 1, 2, 3$, for the object A as well as the spatial dimensions x_i for the observer. The vertical axis shows both time dimensions with notation conform Eq. (2), so $x_4 = c\tau$. Due to object A being at rest, relative to the observer, the axes overlap. The circle is just a tool to better show the rotation that will be depicted in Fig. 2.

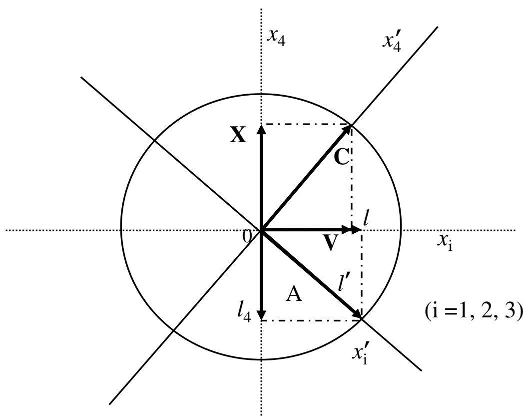

Figure 2: Object A in motion relative to observer. The dimensional axes of object A have rotated relative to the observer.

Definitions are as follows:

- Vector **C** indicates the 4D velocity, having magnitude c, of object A.
- Vector **V**, of magnitude v, and **X**, of magnitude χ, are the projections of this velocity **C** on, respectively, the spatial dimensions and the proper time dimension of the observer.
- l' indicates the proper length of object A in the spatial direction x'_i in the rest frame of object A (in this example l' is also set to c).
- l and l_4 are, respectively, the projections of this proper length on the spatial dimensions and the proper time dimension of the observer.

In Fig. 2, object A moves with speed v relative to the observer. This leads to a relative rotation of dimensions x'_4 and x'_i such that **V** is the projection of the original 4D velocity **C** of object A on the x_i axis of the observer at rest. The situation is examined at the instant where $x_i = x'_i = x_4 = x'_4 = 0$.

The Lorentz transformation equation for x is

$$x' = \gamma(x - vt) \tag{7}$$

where

$$\gamma = 1/\sqrt{1 - v^2/c^2} \tag{8}$$

but this factor can also be written as

$$\gamma = c/\sqrt{c^2 - v^2} = c/\chi \tag{9}$$

leading to

$$x' = c(x - vt)/\chi \tag{10}$$

At $t = 0$, the length of object A will be contracted, as measured by the observer, according to

$$x = x'\chi/c \tag{11}$$

so the contraction of length l can be written as

$$l = l'\chi/c \tag{12}$$

which shows that l, as measured by the observer at rest, is indeed the goniometric projection of the proper length l' on the x_i axis.

Arrow l_4 is the projected 'length' component of the moving object A on the proper time axis x_4 of the observer as a result of the rotation of the dimension x'_i. This length is the manifestation of the difference in proper time (the non-simultaneity) between the endpoints of object A in motion according to the Lorentz transformation equation for time:

$$t' = \gamma(t - vx/c^2) \tag{13}$$

and can be interpreted as a rotation 'out of space' of the proper length l' towards the negative axis of x_4. At $t = 0$ the proper-time difference between tail and head of arrow l will be

$$t' = -\gamma v l/c^2 = -lv/c\chi \tag{14}$$

From $l = l'\chi/c$ and $l_4 = l'v/c$ it follows that

$$l_4 = -ct' \tag{15}$$

which confirms that l_4 represents the proper-time difference in object A. The factor c results from the choice of units for space and time.

Summarizing, from the perspective of the observer, the proper length l' of object A is decomposed in the components l and l_4 according to:

$$l'^2 = l^2 + l_4^2 \tag{16}$$

and so is also the 4D speed c of the object decomposed in the components χ and v:

$$c^2 = \chi^2 + v^2. \tag{17}$$

Equation (16) thus combines Eqs. (7) and (13) into a single Pythagorean equation in four dimensions.

4 Relativistic Addition of Velocities

Since the Lorentz transformation is still valid in the Euclidean approach and those transformations build a group, the derivation of the addition equation does not differ from the classical Minkowski-based approach. The equation therefor remains the classical one.

Let:

- v be the speed of an observer with rest frame x' as measured by an observer with rest frame x.
- w be the speed of a third object as measured by the observer with rest frame x.
- u be the speed of that same object but now as measured by the observer with rest frame x'.

When the directions of u, v, and w are parallel, the classical relation between them is:

$$w = \frac{u + v}{1 + uv/c^2} \tag{18}$$

5 Relativistic Doppler Effect

Using the identity $\chi = \sqrt{c^2 - v^2}$ for the time-speed variable in the wavelength equation for the relativistic Doppler effect

$$\lambda' = \lambda_0 \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{19}$$

simplifies this expression to

$$\lambda' = \lambda_0 (c + v)/\chi \tag{20}$$

It is possible to identify the individual contributions of the factors v and χ to the total Doppler effect by considering $\chi = c$ (which isolates the effect of the spatial speed) and $v = 0$ (which isolates the effect of the time-speed).

Setting $\chi = c$ results in:

$$\lambda'_v = \lambda_0 (1 + v/c) \tag{21}$$

which is the regular equation for the acoustic Doppler effect with moving source and stationary receiver. Setting $v = 0$ results in:

$$\lambda'_\chi = \lambda_0 c/\chi \tag{22}$$

which simply makes the photon's frequency proportional to the time-speed of the emitting particle.

The relativistic Doppler effect can thus be interpreted as a combination of the normal 'acoustic' Doppler effect in space and a frequency shift that results from the lower time-speed.

6 Mass, Energy and Momentum

It follows naturally that the Lorentz invariant $m_0 c$ (m_0 is the rest mass) of a moving object with speed v can be decomposed in

$$m_0^2 c^2 = m_0^2 \chi^2 + m_0^2 v^2 \tag{23}$$

which, using the identities $E = \gamma m_0 c^2$ and $p = \gamma m_0 v$, is equivalent to the classical equation

$$E^2/c^2 = m_0^2 c^2 + p^2 \tag{24}$$

E being the total energy and p being the spatial momentum. Note the existence of only components in the form of momentum in Eq. (23).

7 Euclidean Four-Vectors

The traditional Minkowski line element with metric $(+1, -1, -1, -1)$ is:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{25}$$

where $ds = cd\tau$. Four-vectors with the Euclidean metric $(+1, +1, +1, +1)$ as used in the previous Sections use the 4D velocity of the moving object and 4D Euclidean distances as invariants, which is in fact the essence of Eq. (2):

$$c^2 = v_1^2 + v_2^2 + v_3^2 + \chi^2 \tag{26}$$

Multiplication with $dt^2 = dx_5^2$ yields (recall that $\chi = cd\tau/dt$):

$$c^2dt^2 = dx_1^2 + dx_2^2 + dx_3^2 + c^2d\tau^2 \tag{27}$$

where the factors $ds^2 = c^2d\tau^2$ and c^2dt^2 from Eq. (25) have switched roles.

The Euclidean metric thus gives rise to four-vectors for position, velocity and energy/momentum:

Euclidean	Minkowskian
$(x_1, x_2, x_3, c\tau)$	(x_1, x_2, x_3, ct)
(v_1, v_2, v_3, χ)	$\gamma(v_1, v_2, v_3, c)$
$(m_0v_1, m_0v_2, m_0v_3, m_0\chi)$	$(p_1, p_2, p_3, E/c)$

Equation (27) is not really new. It is merely Eq. (25) written in a different form, with as a main input the definition of χ, being the time-speed of an object as measured by an observer at rest, which has three effects:

- It creates a new invariant c, being the universal magnitude of the 4D velocity of an object.
- It provides a Euclidean basis for the definition of vectors in the direction of the time dimension.
- It enables these new vectors to be summed with existing vectors in the spatial dimensions.

In general, the new Euclidean four-vectors can be derived from the Minkowski four-vectors by using the time component in the Minkowski four-vector as the invariant (the vector sum) for the new four-vector. It is essentially doing Pythagoras "the other way around", *i.e.*, calculating the hypotenuse from the rectangular sides, instead of calculating a rectangular side from the hypotenuse and the other rectangular side (refer to [7] for a detailed treatment of Minkowski and Euclidean four-vectors).

References

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[6] Brian R. Greene, **The elegant universe**, page 391 note 5 (W.W. Norton & Company New York-London, 1999).

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