# General relativity in Euclidean terms

By R. d'E. Atkinson

*Royal Greenwich Observatory, Herstmonceux*

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The relativistic equations for the deflexion of light, the motion of a particle, and the red shift of spectral lines, in the neighbourhood of a single stationary mass, are rigorously derived on the basis of a strictly Euclidean space and an independent time. Only two *ad hoc* assumptions are needed, in addition to two very obvious extensions of the special theory; one of these assumptions is already familiar, but the other, involving the mass of a stationary test particle, is believed to be new. The particle equations are derived from a Lagrangian in the usual way. Expressions for the kinetic and potential energies are also readily obtained. It is shown (by what is believed to be a new argument) that matter with an infinite Young's modulus cannot exist, and the fact that actual measuring rods may therefore be affected by tidal forces, even when they are 'unconstrained', is considered. It is shown that in principle observations in the solar system should be made in a time system which is not that in which the clocks of distant observatories are synchronized at present; the difference is below the present errors of the best time signals, but not very much below. A rigorous expression is derived for the numerical value of the radial co-ordinate $r$, in terms of quantities directly observable by the crew of a space-ship (of negligible mass) moving in a circular orbit at the appropriate circular velocity. Further progress along these lines will depend on their extension to the two-body problem.

It is widely recognized that the literature on general relativity which has grown up during the last 45 years contains a number of misconceptions, from which some of the most eminent pioneers have not been entirely exempt. The 'spinning disk', for example, has been rather imperfectly treated in very many cases, and 'ideally incompressible' matter has been postulated although any matter in which the velocity of sound exceeds $c$ is necessarily non-relativistic. Even today, there is no universal agreement on the validity of some of the arguments that have appeared, and have then been publicly or privately challenged.

This situation might almost have been predicted: it is easy to extend the formal algebra of three-dimensional analysis to four dimensions and more, and reasonably easy (at least in the simpler cases) to generalize it also from Euclidean space to non-Euclidean; but it is extremely difficult to balance this algebraic analysis with any sort of conceptual approach, once the familiar landmarks of everyday 'physical' thinking are lost. There is a very real gap between those who appreciate the beauty and symmetry of the formal mathematics so keenly that they may even deny any need for 'visualization' altogether, and those on the other hand whose natural interests and abilities lie in the field of specific observation and measurement, but who are, almost as a consequence, overwhelmed by the *conceptual* difficulties of four dimensions and curved space, and so cannot consider the fundamental aspects of relativity with the appropriate confidence.

A direct bridging of the gap does not seem easy to achieve from either side. But it might be circumvented, if one could carry through an entirely fresh derivation of the equations of general relativity, within the framework of a rigorously Euclidean space and an independent 'Newtonian' time. This has proved, on trial, to be quite practicable. And the result is that the experimentally minded can now, it may be hoped, find a firm path through to the accepted equations, without encountering the old conceptual difficulties at all, while the mathematically inclined, though they may temporarily sacrifice everything that seems to them most attractive about the theory, will see a quite straightforward road back to their own familiar ground at the end of the excursion. Indeed, if the crucial experimental results predicted by the general theory, together with the increase of inertial masses at high speeds (which was already known when that theory appeared), had been precisely known from observation before the Michelson–Morley experiment had been performed, it is possible that the entire theory might first have developed along the lines now to be followed, once the results of that experiment had been explained by the contraction hypothesis of Fitzgerald.

Some steps along this road were taken quite early. Eddington, for example (1920, p. 54), discussed the deflexion of light in terms of a variable 'refractive index' in a space which was, itself, Euclidean. It is true that he worked only to the first approximation in this, and this restriction tends to obscure the question whether Euclidean co-ordinates are or are not completely satisfactory for a rigorous treatment; but it appears almost certain that his approximations were made only for algebraic convenience, and that he recognized the Euclidean approach to be logically permissible, even though the relativistic one which he later substituted seemed to him more attractive mathematically.

There are in fact two effective, but mutually exclusive, lines of argument, of which only one has been well explored as yet. It is possible, on the one hand, to postulate that the velocity of light is a universal constant, to define 'natural' clocks and measuring rods as the standards by which space and time are to be judged, and then to discover from measurement that space-time, and space itself, are 'really' non-Euclidean; alternatively, one can *define* space as Euclidean and time as the same everywhere, and discover (from exactly the same measurements) how the velocity of light, and natural clocks, rods, and particle inertias 'really' behave in the neighbourhood of large masses. There is just as much (or as little) content for the word 'really' in the one approach as in the other; provided that each is self-consistent, the ultimate appeal is only to convenience and fruitfulness, and even 'convenience' may be largely a matter of personal taste; but neither the fruitfulness of the Euclidean treatment nor its self-consistency can be tested until it has been adequately developed.

The present discussion deals only with the case of a single mass at rest at the origin, and with the consequences of Einstein's original equations $G_{\mu\nu} = 0$. It may, however, be hoped that it will point the way to a similar treatment for two, or more, masses. Presumably, in that case, a stage will come where approximations become unavoidable; the present derivations, however, are rigorous.

It will be convenient first to summarize the relativity results which must be reached, if the proposed plan is to claim to have succeeded. Most of the equations are readily available, free of all approximation, in the form they take when the line-element in spherical polar co-ordinates $(r_1, \theta, \phi)$ is the 'Schwarzschild' one given by

$$ds^2 = \gamma dt^2 - \gamma^{-1} dr_1^2 - r_1^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

where $\gamma \equiv 1 - 2m/r_1$. For our purpose, however, it seems better to use the 'isotropic' solution; this is less familiar, probably because it is often a little less elegant, and some of the 'isotropic' equations may even be hard to find at all in the literature. The derivation of all of them will therefore be sketched, for completeness; but it is not essential to the main purpose of the paper. Essentially, these derivations may be omitted, and it may be taken for granted that the accepted relativity position, in so far as it can be expressed by equations rather than by a philosophy, is in fact rigorously stated by equations (12), (15), (16), (17) and (20). (It is convenient to treat (12) separately; but it is, of course, the intergal of energy, and can be directly derived from (15) and (16).)

The isotropic equations may be obtained from the equivalent 'Schwarzschild' ones by means of the substitution

$$r_1 = r(1+\psi)^2, \tag{2}$$

where $\psi \equiv m/2r$. In addition to making this substitution, we re-introduce the constant $c$ ($\approx 3 \times 10^{10}$ cm/s) explicitly, i.e. we replace $dt^2$ in (1) by $c^2 dt^2$. This equation then becomes

$$ds^2 = \left(\frac{1-\psi}{1+\psi}\right)^2 c^2 dt^2 - (1+\psi)^4 [dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{3}$$

and the exact meaning of $\psi$ is $fM/2rc^2$ where $M$ is the central mass in grams and $f$ is the constant of gravitation in c.g.s. ($\approx 6{\cdot}67 \times 10^{-8}\, g^{-1}\, cm^3\, s^{-2}$). The four geodesic equations, which describe the motion of a test particle, may be obtained from their Schwarzschild equivalents, given, for example, by Eddington (1923, pp. 85–6), in the same way. If preferred, they may of course also be obtained directly, by inserting the $g_{\mu\nu}$ of (3) into the four equations of a geodesic,

$$\frac{d^2 x_\alpha}{ds^2} + \sum_{\lambda,\mu,\nu}^{1-4} \tfrac{1}{2} g^{\alpha\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right) \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0 \tag{4}$$

($\alpha = 1, \ldots, 4$); taking $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$, $x_4 = t$, we have

$$g_{11} = -(1+\psi)^4, \quad g_{22} = -(1+\psi)^4 r^2, \quad g_{33} = -(1+\psi)^4 r^2 \sin^2\theta,$$
$$g_{44} = (1-\psi)^2 c^2/(1+\psi)^2, \quad g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu$$

and therefore $g^{11} = 1/g_{11}$, etc.; and since $d\psi/dr = -\psi/r$ the resulting equations will be found to be

$$\frac{d^2 r}{ds^2} - \frac{2\psi}{r(1+\psi)}\left(\frac{dr}{ds}\right)^2 - r\frac{1-\psi}{1+\psi}\left(\frac{d\theta}{ds}\right)^2 - r\frac{1-\psi}{1+\psi}\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 + \frac{2\psi c^2}{r}\frac{1-\psi}{(1+\psi)^7}\left(\frac{dt}{ds}\right)^2 = 0, \tag{5}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{1-\psi}{1+\psi}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0, \tag{6}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{1-\psi}{1+\psi}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0, \tag{7}$$

and
$$\frac{d^2t}{ds^2}+\frac{4\psi}{r}\frac{1}{(1+\psi)(1-\psi)}\frac{dr}{ds}\frac{dt}{ds}=0. \tag{8}$$

As in the Schwarzschild case, if we choose axes so that $\theta=\frac{1}{2}\pi$ and $d\theta/ds=0$ initially, (6) yields $d^2\theta/ds^2=0$ and so $\theta=\frac{1}{2}\pi$ permanently; we can thus simplify (3), (5) and (7) by putting $\theta=\frac{1}{2}\pi$ and $d\theta/ds=0$ there also.

The observational consequences which follow from these equations, in the relativistic treatment, are all ultimately expressed in equations from which d$s$ has been eliminated. For our purpose, it is simplest to undertake this elimination at once, and this can be done without any real loss of content. Equation (8) integrates directly, giving
$$\left(\frac{1-\psi}{1+\psi}\right)^2 c\frac{dt}{ds}=B, \tag{9}$$

where $B$ is a (dimensionless) constant;* and since we can rewrite (3) as
$$\left(\frac{ds}{dt}\right)^2=\left(\frac{1-\psi}{1+\psi}\right)^2c^2-(1+\psi)^4v^2, \tag{10}$$

where
$$v\equiv(dr^2+r^2d\phi^2)^{\frac{1}{2}}/dt \tag{11}$$

(i.e. $v$ is the 'co-ordinate velocity'), we obtain instead of (9) and (10),
$$\frac{1-\psi}{1+\psi}\left[1-\frac{(1+\psi)^6}{(1-\psi)^2}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}=B. \tag{12}$$

To eliminate d$s$ also from (5) and (7) we use the identity
$$\frac{d^2}{ds^2}\equiv\left(\frac{dt}{ds}\right)^2\frac{d^2}{dt^2}+\left(\frac{d}{dt}\frac{dt}{ds}\right)\frac{dt}{ds}\frac{d}{dt} \tag{13}$$

or, with (9),
$$\frac{d^2}{ds^2}=\left(\frac{dt}{ds}\right)^2\left[\frac{d^2}{dt^2}-\frac{4\psi}{r}\frac{1}{(1+\psi)(1-\psi)}\frac{dr}{dt}\frac{d}{dt}\right]; \tag{14}$$

since $dt/ds$ never vanishes, the new equations are then
$$\frac{d^2r}{dt^2}-\frac{2\psi}{r}\frac{3-\psi}{(1+\psi)(1-\psi)}\left(\frac{dr}{dt}\right)^2-r\frac{1-\psi}{1+\psi}\left(\frac{d\phi}{dt}\right)^2+\frac{2\psi c^2}{r}\frac{1-\psi}{(1+\psi)^7}=0 \tag{15}$$

and
$$\frac{d^2\phi}{dt^2}+\frac{2}{r}\frac{1-4\psi+\psi^2}{(1+\psi)(1-\psi)}\frac{dr}{dt}\frac{d\phi}{dt}=0. \tag{16}$$

These two equations involve only two spatial co-ordinates and the time; together with (12), they contain all the essential statements that relativity makes about the motion of a test particle in the neighbourhood of a single stationary mass. Part of our task, therefore, will be to make such arbitrary assumptions, in the framework of a strictly Euclidean space and an independent time, as will lead rigorously to these equations. One might expect that the assumptions, since they will in any case be of an *ad hoc* nature, would include some arbitrary law of gravitation differing a little from the Newtonian, and perhaps also some modification of the laws of

* It is perhaps a little confusing that Eddington used $c$ for this integration-constant. The notation $1+\epsilon$ instead, introduced at (34) below, will be found safer, at least if one wishes to re-introduce the velocity of light explicitly.

dynamics as such; but the path we actually take involves instead assumptions onl about the velocity of light and the mass of a test particle. Nothing either new or ev old is assumed (at least expressly) about gravitation, and we shall find that the onl modifications to the accepted laws of dynamics are a couple of very obvio extensions of formulae already familiar from the special theory.

In addition to reaching equations (12), (15) and (16) for the motion of a partich we have also to reach the rigorous relativity formulae for the deflexion of ligh and for the red shift. (Not, of course, the red shift for a distant nebula; we ar concerned here with the 'solar' or 'white dwarf' red shift.) This effect, if expresse as a 'red ratio', $\nu_r/\nu$ say, is given in the isotropic system by

$$\frac{\nu_r}{\nu} = \frac{1-\psi}{1+\psi}; \tag{17}$$

$\nu_r$ is here the frequency, when its radial co-ordinate is $r$, of a natural system whos frequency is $\nu$ when $r = \infty$.

A rigorous equation for the deflexion is given by Eddington (1923, p. 90) in the forn

$$\mathrm{d}^2u_1/\mathrm{d}\phi^2 + u_1 = 3mu_1^2, \tag{18}$$

where $u_1 \equiv 1/r_1$, i.e. in terms of the Schwarzschild co-ordinate system. Multiplyin by $2\mathrm{d}u_1$, and integrating, we have

$$u_1^2 + (\mathrm{d}u_1/\mathrm{d}\phi)^2 = 2mu_1^3 + A, \tag{19}$$

where $A$ is a constant. If we transform this by (2) as before, and write $u \equiv 1/r$ (so that we also have $2\psi = mu$) we obtain

$$u^2 + \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = A\frac{(1+\psi)^6}{(1-\psi)^2}. \tag{20}$$

Equations (17) and (20) thus complete the challenge which faces us.

We begin by deriving (20). In the relativity treatment, the 'co-ordinate velocity' of light at any point is obtained by setting $\mathrm{d}s = 0$ in (10); $v$ then refers to a light-pulse. For our purposes, this co-ordinate velocity is simply 'the' velocity, $c_r$ say, and we therefore adopt, as our first *ad hoc* postulate,

$$c_r = \frac{1-\psi}{(1+\psi)^3}c, \tag{21}$$

where $c_r$ is to be understood in the most elementary and straightforward sense, as the actual velocity of light in Euclidean space, at the distance $r$ from the origin.

If, now, $P$ is the perpendicular from the origin onto the tangent at some point on a light-ray, then, as in Eddington's earlier treatment (1920, p. 54), since the 'refractive index' $c/c_r$ is stratified in spherical shells about the origin, Snell's law may be written

$$Pc/c_r = \text{const.} \tag{22}$$

for all points on one ray. And since

$$\frac{1}{P^2} = u^2 + \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 \tag{23}$$

we have (20) at once.

The constant $A$ is readily evaluated. At the point of closest approach, $u = u_0$ say, we have $du/d\phi = 0$; accordingly

$$A = \frac{(1-\psi_0)^2}{(1+\psi_0)^6} u_0^2 \tag{24}$$

and (20) is therefore

$$\frac{(1-\psi)^2}{(1+\psi)^6}\left[u^2+\left(\frac{du}{d\phi}\right)^2\right] = \frac{(1-\psi_0)^2}{(1+\psi_0)^6} u_0^2. \tag{25}$$

To the first order in $mu$ (or $2\psi$) this is

$$(1-4mu)\,[u^2+(du/d\phi)^2] \approx (1-4mu_0)\,u_0^2 \tag{26}$$

and, as may readily be verified, this is satisfied (to the same accuracy) by

$$u \approx u_0 \cos\phi(1-2mu_0)+2mu_0^2 \tag{27}$$

if we take $\phi = 0$ when $u = u_0$. This is equivalent to

$$x \approx r_0+2m-(2m/r_0)\sqrt{(x^2+y^2)} \tag{28}$$

(where we have written $r\cos\phi = x$, $r = \sqrt{(x^2+y^2)}$), i.e. a hyperbola whose asymptotes are

$$x \approx r_0+2m \pm (2m/r_0)\,y. \tag{29}$$

The angle, $D$, between the asymptotes is the total deflexion, so that

$$D \approx 4m/r_0 \tag{30}$$

in agreement (to this order) with the Schwarzschild result; in particular, there is of course no discrepancy by the factor $\frac{1}{2}$ which, as is well-known, characterized the primitive 'Newtonian' treatment.

We turn, next, to the task of deriving the 'geodesic' equations. By analogy with the special theory, we take it for granted that the (inertial) mass of a moving test particle will be greater than that of the same particle at rest, at the same place, by the factor $(1-v^2/c_r^2)^{-\frac{1}{2}}$, i.e. the familiar expression, but with the local value for the velocity of light. (Clearly $c_r$ must be the limit, at that point, above which no velocity can go; other more complicated formulae could also guarantee this, but there is no need to try them.) Accordingly, if the mass of the test particle, when at rest at $r$, is $\mu_r$, its mass when it is in motion with the velocity $v$ will be taken to be $\mu_r(1-v^2/c_r^2)^{-\frac{1}{2}}$; we assume also that its total energy, $H$, is then given by

$$\begin{aligned} H &= \mu_r(1-v^2/c_r^2)^{-\frac{1}{2}}c_r^2 \\ &= \mu_r c^2\frac{(1-\psi)^2}{(1+\psi)^6}\left[1-\frac{(1+\psi)^6}{(1-\psi)^2}\frac{v^2}{c^2}\right]^{-\frac{1}{2}}. \end{aligned} \tag{31}$$

If $H$ is constant, we may write

$$H = \mu c^2(1+\epsilon) \tag{32}$$

where $\mu$ and $\epsilon$ are constants; $\mu$ has evidently the dimensions of a mass. Thus

$$\frac{\mu_r}{\mu}\frac{(1-\psi)^2}{(1+\psi)^6}\left[1-\frac{(1+\psi)^6}{(1-\psi)^2}\frac{v^2}{c^2}\right]^{-\frac{1}{2}} = 1+\epsilon. \tag{33}$$

This equation will be identical with (12) if (as is naturally permissible) we adju the two arbitrary dimensionless constants so that

$$B = 1+\epsilon \tag{3}$$

and if, in addition, $$\mu_r/\mu = (1+\psi)^5/(1-\psi); \tag{3}$$

we therefore adopt (35) as a second *ad hoc* postulate, in addition to (21), so th (31) becomes

$$H = \mu c^2 \frac{1-\psi}{1+\psi}\left[1-\frac{(1+\psi)^6}{(1-\psi)^2}\frac{v^2}{c^2}\right]^{-\frac{1}{2}};$$

we can also rewrite (12) as

$$\frac{1-\psi}{1+\psi}\left(1-\frac{v^2}{c_r^2}\right)^{-\frac{1}{2}} = 1+\epsilon. \tag{3}$$

Putting $\psi = 0$, i.e. $r = \infty$, we see from (35) that $\mu$ is the value of $\mu_r$ when $r = \infty$ and since $\mu_r$ refers in any case to a stationary particle, $\mu$ is the test particle's ma when it is stationary at infinity.

We now assume that we can construct, and use, a Lagrangian in the standar way. We therefore write

$$\begin{aligned} L &= \mathbf{p}\,.\,\mathbf{v} - H \\ &= \mu\left(1-\frac{v^2}{c_r^2}\right)^{-\frac{1}{2}}\left[v^2\frac{(1+\psi)^5}{1-\psi} - c^2\frac{1-\psi}{1+\psi}\right] \\ &= -\mu c^2 \frac{1-\psi}{1+\psi}\left(1-\frac{v^2}{c_r^2}\right)^{+\frac{1}{2}} \\ &= -\mu c^2 \frac{1-\psi}{1+\psi}\left[1-\frac{(1+\psi)^6}{(1-\psi)^2}\frac{\dot r^2+r^2\dot\phi^2}{c^2}\right]^{\frac{1}{2}} \end{aligned}$$

(38), (39), (40)

together with $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0, \tag{41}$$

where $q$ denotes, as usual, a generalized co-ordinate; and we have to show that these equations reduce to (15), and to (16). It is convenient to note the identity

$$\frac{\mathrm{d}}{\mathrm{d}r}[(1+\psi)^m(1-\psi)^n] \equiv (1+\psi)^m(1-\psi)^n\frac{\psi}{r}\frac{(n-m)+(n+m)\psi}{(1+\psi)(1-\psi)} \tag{42}$$

which finds repeated application in what follows.

Taking first the $\phi$-co-ordinate, we have, since $\partial L/\partial\phi = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot\phi} = 0 \tag{43}$$

or $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1-\psi}{1+\psi}\left(1-\frac{v^2}{c_r^2}\right)^{-\frac{1}{2}}\frac{(1+\psi)^6}{(1-\psi)^2}r^2\dot\phi\right] = 0. \tag{44}$$

With (37), this gives $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(1+\psi)^6}{(1-\psi)^2}r^2\dot\phi\right] = 0 \tag{45}$$

i.e. $$\ddot\phi + \frac{2}{r}\frac{1-4\psi+\psi^2}{(1+\psi)(1-\psi)}\dot r\dot\phi = 0 \tag{46}$$

which is exactly the '$\phi$-geodesic', (16). This equation, as is of course to be expected, is simply a statement of the conservation of angular momentum; even without help from (43), we would naturally write

$$\mu_r\left(1-\frac{v^2}{c_r^2}\right)^{-\frac{1}{2}} r^2\frac{d\phi}{dt} = \text{const.}, \tag{47}$$

which is the same as (44). Thus our assumptions have now given us the deflexion of light, and also have given expressions for the conservation of energy and of angular momentum which agree exactly with the appropriate geodesic equations of general relativity.

Already at this stage one advantage of the new point of view has emerged; the physical significance of (47) is very much clearer than that of the integral of (7), namely (when $\cot\theta = 0$)

$$(1+\psi)^4 r^2(d\phi/ds) = \text{const.}, \tag{48}$$

or, as it usually appears,

$$r_1^2(d\phi/ds) = h. \tag{49}$$

Part of Eddington's comment (1923, p. 87) on the difference between (49) and the familiar Newtonian expression

$$r^2(d\phi/dt) = h \tag{50}$$

was as follows: 'the difference between $ds$ and $dt$ [in our notation $c\,dt$] is equally insignificant, even if we were sure what is meant by $dt$ in the Newtonian theory'; and while it is perhaps a little unfair to quote this nowadays, there may be some who would still accept it. In point of fact, although there is, as we shall see more clearly below, an ambiguity (but no lack of rigour) in the choice between $r_1$, $r$, and a whole family of alternative heirs of the Newtonian radius vector, there is no doubt at all about $t$; both in Newtonian mechanics and in general relativity, $t$ (though it may of course appear in the $g_{\mu\nu}$) is independent of the spatial co-ordinates in exactly the same way as they are independent of each other; any observer stationary in the system in question will record his observations in terms of his three spatial co-ordinates and this time. 'Simultaneity' for such an observer is, by definition, the result of putting $t = t_0 =$ const. for all $r, \theta, \phi$. Neither theory regards natural clocks as necessarily keeping this time everywhere, but the observer must be supposed to be able (at least in a static system) to station suitable clocks anywhere, and to *regulate them as necessary* until they do all go at the same rate as his standard clock, and so show this time. (For our purpose, the problem of synchronizing them does not arise, so long as we are concerned only with $dt$, but in principle the observer is supposed to be able to do this also.) The difference between (49) and (50) is not due to any requirement that the independent variable of particle dynamics ought now to be time measured in some new way—$ds$, indeed, is not even eligible at all, since it is not independent of $r$; the difference between (47) and (50) arises entirely, and quite naturally, from the variability of the test particle's mass, $\mu_r(1-v^2/c_r^2)^{-\frac{1}{2}}$; this variability is due partly to its position and partly to its speed. If we now reverse the whole argument, we may say that general relativity appears to imply our assumption (35) just as it implies (21); however, (35) does not seem to have been given explicitly in any previous work.

Whitehead (1922, p. 102) does indeed give in effect the first-order approximation to it; but his derivation begins with a treatment of the many-body problem (and seems difficult even apart from this) and some of his conclusions also differ from Einstein's. The rigorous form (35) is tied to a definite choice of the isotropic system (as opposed to the Schwarzschild one and all other equivalent ones), and Whitehead's treatment is avowedly approximate.

For the $r$-component of the geodesic, we have, first

$$\frac{\partial L}{\partial \dot{r}} = \mu c^2 \frac{1-\psi}{1+\psi}\left(1-\frac{v^2}{c_r^2}\right)^{-\frac{1}{2}} \frac{(1+\psi)^6}{(1-\psi)^2}\frac{\dot{r}}{c^2} \tag{51}$$

$$= \mu(1+\epsilon)\frac{(1+\psi)^6}{(1-\psi)^2}\dot{r}. \tag{52}$$

by (37). Differentiating, with the help of (42), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = \mu(1+\epsilon)\frac{(1+\psi)^6}{(1-\psi)^2}\left[\ddot{r} - \frac{2\psi}{r}\frac{4-2\psi}{(1+\psi)(1-\psi)}\dot{r}^2\right]. \tag{53}$$

Similarly, after some reduction, we have

$$\frac{\partial L}{\partial r} = -\mu(1+\epsilon)\frac{(1+\psi)^6}{(1-\psi)^2}\left[\frac{2\psi}{r}\frac{1-\psi}{(1+\psi)(1-\psi)}\dot{r}^2 - \frac{1-\psi}{1+\psi}r\dot{\phi}^2 + \frac{2\psi c^2}{r}\frac{1-\psi}{(1+\psi)^7}\right]. \tag{54}$$

Thus, when $q$ stands for $r$, (41) becomes

$$\ddot{r} - \frac{2\psi}{r}\frac{3-\psi}{(1+\psi)(1-\psi)}\dot{r}^2 - r\frac{1-\psi}{1+\psi}\dot{\phi}^2 + \frac{2\psi c^2}{r}\frac{1-\psi}{(1+\psi)^7} = 0 \tag{55}$$

which is identical with (15). Since we have now derived (12), (15) and (16), we have derived the rigorous theory of the advance of perihelion (and all other 'relativistic perturbations' such as the lunar ones, in so far as they can fairly be derived from a one-body solution at all), strictly within our Euclidean framework.

As is well known, attempts were made very early to explain the perihelion advance in this framework, simply by the increase of mass with velocity as given by the special theory, i.e. with $c$ instead of $c_r$ and making no allowance for $\mu_r$. The present analysis shows that although these attempts were bound to fail, they were in fact much more nearly appropriate than has generally been supposed.

The algebra leading to (55) may possibly appear a little cumbrous, but it is an instructive demonstration of the power of the Lagrangian method. The derivation of (47) did not need the full Lagrangian panoply at all; it reduced to an apparent triviality, as it so often does. But the radial equation is more difficult; even though the right-hand side of (51) is plainly the radial component of momentum, the attempt to get further by defining force as rate of change of momentum seems beset by pitfalls. It will be noted, for example, that terms in $\dot{r}^2$, all of them small by comparison with the 'Newtonian' ones, arise both in (53) and in (54).

There is, however, no difficulty about getting explicit expressions for the kinetic and potential energies directly. The kinetic energy is naturally, as in the special theory,

$$K = \mu_r c_r^2[(1-v^2/c_r^2)^{-\frac{1}{2}} - 1] \tag{56}$$

$$= \mu c^2[1+\epsilon-(1-\psi)/(1+\psi)]$$

$$= \mu c^2[\epsilon + 2\psi/(1+\psi)] \tag{57}$$

and $\mu c^2 \epsilon$ is thus the kinetic energy of the test particle at infinity. It is also clear from (37) that if $\epsilon = 0$, $v = 0$ when $r = \infty$, so that this is the 'quasi-parabolic' case; if $\epsilon < 0$, $v$ is imaginary when $r = \infty$, i.e. the 'quasi-elliptical' case. The 'velocity of escape', $v_e$, obtained by putting $\epsilon = 0$ and $r \neq \infty$ in (37), is

$$\begin{aligned} v_e &= \frac{1-\psi}{(1+\psi)^4} 2c\sqrt{\psi} \\ &= \frac{1-\psi}{(1+\psi)^4}\sqrt{\left(\frac{2fM}{r}\right)}. \end{aligned} \tag{58}$$

The potential energy, if we do not include the rest-mass energy, is

$$\begin{aligned} U &= H - K - \mu c^2 \\ &= -\mu c^2 \, 2\psi/(1+\psi) \end{aligned} \tag{59}$$

and vanishes at infinity; if we prefer to include the rest mass, it is of course

$$U_1 = \mu c^2 (1-\psi)/(1+\psi). \tag{60}$$

The fact that the potential energy (per unit particle mass) is definitely not $-2\psi c^2$ (i.e. $-fM/r$) underlines a danger which seems to have been ignored in much of the literature. It is quite common to find $-m/r$ spoken of, simply, as 'the potential', without any inquiry as to what the exact formula for $U$ may be, in the particular system in use. The practice is numerically harmless, if one is only working to the first order in $m/r$, but it could lead to error if it were taken rigorously. (In general, $U$ does not satisfy $\nabla^2 U = 0$ in empty space, and one cannot resort to spherical harmonics without further consideration.)

In the case of a circular orbit ($\ddot{r} = \dot{r} = 0$), (54) gives

$$(r\dot{\phi})^2 = v_c^2 = \frac{2\psi c^2}{(1+\psi)^6} = \frac{2\psi}{(1-\psi)^2} c_r^2 \tag{61}$$

so that (37) becomes

$$1 + \epsilon = \frac{(1-\psi)^2}{(1+\psi)(1-4\psi+\psi^2)^{\frac{1}{2}}}; \tag{62}$$

expanding to the first power of $\psi$ we see that in a circular orbit

$$\epsilon \approx -\psi \tag{63}$$

and so $K \approx \mu c^2 \psi/(1+\psi) \approx -\frac{1}{2}U$, approximately as in the Newtonian case.

We may also expand (55) to one higher order than the Newtonian. We have, rigorously,

$$\ddot{r} - r\dot{\phi}^2 + \frac{fM}{r^2}\frac{1-\psi}{(1+\psi)^7} = \frac{fM}{r^2c^2}\frac{3-\psi}{(1+\psi)(1-\psi)}\dot{r}^2 - \frac{fM}{rc^2(1+\psi)}r\dot{\phi}^2 \tag{64}$$

or, approximately,

$$\ddot{r} - r\dot{\phi}^2 + \frac{fM}{r^2} \approx \frac{fM}{r^2c^2}\left[\frac{4fM}{r} + 3\dot{r}^2 - (r\dot{\phi})^2\right]; \tag{65}$$

the right-hand side is of course zero in the Newtonian case. The three terms in it will, in general, be all of the same order of magnitude. Equation (65) was given by de Sitter (1916) in essentially this form; his notation is, however, a little difficult, and he did not give (64) explicitly.

It may at first sight appear surprising that Newton's inverse square law should emerge, as the zero approximation, from assumptions about the velocity of light and the mass of a particle in Euclidean space. The reason does not lie in any particular powers of $1+\psi$ and $1-\psi$ that may have been adopted in the formulae for $c_r$ and $\mu_r$, but rather in the nature of $\psi$ itself, and in the way we have (quite naturally) constructed $H$ and $L$. We may in fact repeat the analysis for the more general case in which

$$c_r = (1+\psi)^j (1-\psi)^k c \tag{66}$$

and

$$\mu_r = (1+\psi)^l (1-\psi)^p \mu; \tag{67}$$

equation (55) then becomes

$$\begin{aligned}\ddot{r} - \frac{\psi}{r}\frac{3(k-j)+(p-l)+\{3(k+j)+(p+l)\}\psi}{(1+\psi)(1-\psi)}\dot{r}^2 \\ - \frac{1+(k-j+p-l)\psi+(k+j+p+l-1)\psi^2}{(1+\psi)(1-\psi)} r\dot{\phi}^2 \\ + \frac{\psi c^2}{r}[2(k-j)+(p-l)+\{2(k+j)+(p+l)\}\psi](1+\psi)^{2j-1}(1-\psi)^{2k-1} = 0\end{aligned} \tag{68}$$

and (except in the special case where $2k+p = 2j+l$) the last term will always be $FfM/r^2$ in the first approximation, where $F$ is a constant. [The constant will be unity if $2(k-j)+p-l = 2$; and if we have (from the deflexion) $k-j = 4$, we must then also have $l-p = 6$, i.e. $c_r \approx (1-4\psi)c$ and $\mu_r \approx (1+6\psi)\mu$.]

We have still to derive the red shift. We consider the case of a frictionless flywheel, of mass $\mu$ and radius $a$ when it is at infinity. We suppose it to be given an angular velocity, $\omega$, which is so small that all relativity effects, including those involving cross-products with $v/c$ and $\psi$, are negligible; and we suppose that after this it is never subjected to any further couple. Let it be lowered into the 'gravitational field', brought to (translational) rest there, and then (if we wish) released. Its mass is now $\mu_r$, and we suppose that its radius and angular velocity have become $a_r$ and $\omega_r$, respectively. If its angular momentum is to be conserved, we must have

$$a^2\omega = a_r^2\omega_r(1+\psi)^5/(1-\psi). \tag{69}$$

We consider that such a flywheel, free from all couples, is an 'ideal clock', i.e. that all natural frequencies will behave in the same way as $\omega_r$, so that $\omega_r/\omega$ is the red shift. In order to obtain its value, we do what we have not, so far, done (at least, explicitly): we assert that if a local observer measures the velocity of light, using the actual radius of the disk as his unit of length and the actual frequency as his unit of frequency but considering that they are still $a$ and $\omega/2\pi$ instead of $a_r$ and $\omega_r/2\pi$, he will get the value $c$ instead of $c_r$. Accordingly

$$a\omega c_r = a_r\omega_r c, \tag{70}$$

or

$$a^2\omega^2 = a_r^2\omega_r^2(1+\psi)^6/(1-\psi)^2. \tag{71}$$

Combining this with (69) we have

$$\omega_r/\omega = (1-\psi)/(1+\psi) \tag{72}$$

in agreement with (17). We also have

$$a_r(1+\psi)^2 = a, \tag{73}$$

and since the wheel's radius is not affected by its very slow rotation, all lengths must be supposed to contract in this same way, in a region of strong gravitational potential. No direct test of this effect has ever been proposed, but we shall return to it below.

An alternative proof of (17), which does not rely directly on (70), may be obtained from the conservation of energy as follows. We consider a system consisting initially of a stationary atom, of mass $\mu$, and a quantum of frequency $\nu$, both at infinity. The total energy of this system is of course $H = \mu c^2 + h\nu$. We suppose that the atom absorbs the quantum, so that its rest-mass becomes $\mu + h\nu/c^2$, and that it then falls in to the radius $r$, still in the same excited state, i.e. still with the same rest-mass. Its kinetic energy is now

$$K = (\mu + h\nu/c^2)c^2\, 2\psi/(1+\psi) \tag{74}$$

by (57), since $\epsilon = 0$ in this case. Let this kinetic energy be given up (without, of course, being destroyed) to some surrounding matter by collisions, and immediately after this let the atom return to the unexcited state by emission of a quantum of frequency $\nu_r$. This quantum will escape to infinity still with this frequency (in terms of co-ordinate-time), since we cannot suppose more 'crests' of a wave-train will pass one point than another, per second. (The time system is the same everywhere; thus a frequency change occurring *after* emission, while the train is on its way between two fixed points, would mean that 'crests' had to accumulate indefinitely, in the intervening space. In static conditions, a frequency (unlike a wavelength) cannot be changed en route at all. It can of course be changed by reflexion at a moving mirror; but this changes the space between the two points correspondingly. It can also (Atkinson 1935) be changed if circularly polarized light is passed through a half-wave plate rotating in its own plane, but this is a rather special case and, even so, is not a static one.) The energy of the stationary atom, after emitting the quantum, is $\mu_r c_r^2$, and the total energy is therefore now

$$H = \mu_r c_r^2 + h\nu_r + K. \tag{75}$$

Inserting the values of $K$, $\mu_r$ and $c_r$, we have

$$\mu c^2 \frac{1-\psi}{1+\psi} + h\nu_r + (\mu c^2 + h\nu)\frac{2\psi}{1+\psi} = \mu c^2 + h\nu, \tag{76}$$

i.e.

$$\nu_r = \nu(1-\psi)/(1+\psi) \tag{77}$$

in agreement with the previous result.

It will be noted that this proof has not only avoided appealing to the constancy of the measured velocity of light, but has also not used the conservation of angular momentum (i.e. the invariance of $h$); it has, however, still used (35).

We have now shown that the exact relativity equations can be given an interpretation in terms of a strictly Euclidean framework; but the question may naturally be asked, how any specific values of $r$ are to be precisely identified in practice, if our measuring rods are 'wrong'. This question arises, of course, in relativity theory also; equation (3) is itself rigorous, so that there must be *rigorous* ways of assigning specific values to $r$, and to $\psi$, if it is to have a meaning, but $ds$ is what is actually measured. The first answer certainly must be that since the equations are the same in both cases, any experiment which really can identify an exact value of $r$, for use in the relativity treatment, will also do just the same in the Euclidean one. But the new point of view may possibly suggest other experiments, or other criteria by which to condemn or accept any experiments suggested. And at least it brings out into the open the apparent paradox that although the Schwarzschild co-ordinates, the isotropic ones, and all others of this infinite family, are certainly identical in respect of $\theta$ and $\phi$, and certainly not identical in respect of $r$, nevertheless, each of them is, by itself, strictly Euclidean. (We have not actually demonstrated this; but it can in fact be accepted* that if the refractive index, $c/c_r$, and the mass, $\mu_r$, are suitably modified, the assumptions we have made for the isotropic solution will also be satisfactory for the Schwarzschild one, or for any other of the family, and this includes the assumption of Euclidean space, in each instance.) Before we attack the paradox, or treat any question involving spatial surveys in general, it seems desirable to dispose of the 'ideal incompressible material' which has sometimes been specified for measuring rods that have to be exposed to gravitational or tidal forces.

It is fairly generally agreed, nowadays, that such 'ideal' material cannot exist. The commonest proof relies on the theorem that if the elasticity is infinite, the velocity of sound will be so as well, and yet this must not exceed $c$. However, the bulk modulus (which is the one most usually discussed) raises doubts as to whether the particle density or the proper density (or some other 'density') is really what will determine the velocity of sound; and if we discuss measuring rods it is in any case Young's modulus which should be considered. The following proof (which is believed to be new) avoids all discussion of particular elasticities, and of sound waves; it is slightly simpler to give it in terms of the relativity viewpoint, but either could be used.

If we construct a tetrahedron, by means of six rods joining in all possible ways four non-coplanar points of which no three are collinear, we can vary the length of any rod arbitrarily (within wide limits) and the tetrahedron will adjust itself correspondingly. This is true whether or not the space is Euclidean, and we cannot in general find out anything about the metric of the space by measuring these six lengths alone. If we add a fifth point (avoiding collinear and coplanar cases as before), we can join it to any three of the earlier points similarly, and can then still vary any of the rods without conflict; but as soon as these nine lengths are all definitely settled, the configuration of all five points is fixed, and the length of the one remaining connexion still be to made is already calculable. In general, only one particular length will do, in any particular space. If we calculate this length on the assumption that the rods have been assembled in Euclidean space, and if on trial we find that this last rod will not fit, then the space is not Euclidean, and the magnitude of the misfit will give us one item of information about the actual metric; increasing the number of points, and of cross-connexions, will increase the information which can be obtained. If we now correctly compute and assemble such a structure in Euclidean space, and then move it into non-Euclidean, we have the following mutually exclusive possibilities: (a) the comparison between standard (unconstrained, stationary) measuring rods and the rods of the assembly tells us

* I am indebted to Professor Pryce for pointing this out to me.

that the measured intervals between the five points have changed, so that they no longer fit our computations, and the space is indeed now non-Euclidean; or (*b*) the rods of the assembly are 'ideal incompressible' rods, of which the definition is that the comparison between their lengths and those of unconstrained stationary measuring rods always gives the same answers; in this case, measurement necessarily reveals that the space is still Euclidean although by hypothesis it is not. Thus our supposedly ideal matter is a relativistic impossibility; not only can complex ('over-connected') assemblies of it not be moved from Euclidean space into non-Euclidean, but they also cannot move, in non-Euclidean space, from the point of original assembly at all, except in certain very particular ways (e.g. in a suitable circular orbit if the space happens to be spherically symmetrical). 'Ideal, incompressible matter' would allow us to construct a veritable 'Mahomet's coffin', unsupported, but unable to fall down, and its behaviour would be just as miraculous as it would in the Newtonian case: it would involve a violation of the laws of physics.

We must, therefore, accept the fact that any experiment which is proposed for determining specific values of $r$ by means of measuring rods will have to make do without 'ideal' ones; if tidal forces can stretch or compress them (and they always can, to some extent, if the measuring rods are free), then the change of length will have to be considered, and allowed for if it is appreciable. There are suggested experiments (impracticable, but conceivable) where the effect does not enter, or where it can be arranged to be negligible; but it is generally agreed, nowadays, that it is better, whenever we use any measuring rod at all, to calibrate it by etalon-interferometry, *in situ*.

The calibration can be undertaken if we assume a value for the velocity of light ($c_r$) and also for the frequency of the source emitting it, so that we are sure of the wavelength. (Wavelengths of light *emitted from a local source* obviously contract according to the same formula as 'measuring rod' lengths, i.e.

$$\lambda_r(1+\psi)^2 = \lambda, \tag{78}$$

but those from other sources vary as $c_r/\nu$, not $c_r/\nu_r$.) This rule effectively brings all methods which use measuring rods (in so far as any are really proposed, for the solar system as a whole) into the same class as the methods already familiar to astronomers. The astronomer never measures out space by throwing up large numbers of measuring rods and arranging that they shall all be stationary at the tops of their trajectories at some one specified instant, and shall then just touch end-to-end and mark out some 'over-connected' structure whose properties can be used to discover the metric of space: he observes with suitably disposed theodolites and transit circles, and allows for any refractive index which he believes is present, and for any elapsed light times, and he will have no objection to allowing for frequency changes too, where an actual length is needed, e.g. as the baseline for a geocentric parallax. And we now have the answer to our apparent paradox: any observations we may make for determining specific $r$-values, if serviceable at all, are serviceable whatever (admissible) assumptions we may make; but the conclusions which will be drawn from them will depend on the refractive index used in reducing them (and, in some cases, on the values assumed for natural wavelengths). If we define the refractive index as unity everywhere, we find out that space is non-Euclidean. If we define space as Euclidean, we find that the radial co-ordinate of our Euclidean geometry is $r$ if we reduce all our observations with the refractive index $(1+\psi)^3/(1-\psi)$, and we find that it is $r_1$ (very nearly $r+m$) if we use the anisotropic refractive index appropriate to the Schwarzschild solution. And similarly for any other admissible solution. The different radial co-ordinates should not be regarded as each of them approximately the radius vector of an approximately equivalent Euclidean space; taken in conjunction with the appropriate formulae for reducing observations, any one of them (separately, of course) is rigorously a Euclidean radius vector as it stands. The one we have used here is singled out by the fact that the appropriate refractive index is isotropic, and the convenience of this has in fact been of some help; but it is neither more nor less rigorously Euclidean than the others on that account.

We now consider, from the new viewpoint, a preliminary problem in planetary kinematics which has perhaps not been treated in the general theory. If we have two planets moving in opposite directions round two almost-coincident circular orbits, then near the moment where they are passing each other the observations which either may make of the other can be related, for a short while, just as in the special theory, i.e. by a simple Lorentz transformation; this means that each will consider that the other's clocks are going slow. But certainly when they next meet, half an orbit later, they will agree that the loss has somehow been made up again; by mere symmetry, neither can really lose on the other at all. Since the rate which one 'observes', for a clock moving past one's system, is intimately bound up with the way one reckons 'simultaneity' within one's system, each observer will suspect that he has chosen a method of synchronizing his own clocks which is not really very advantageous for discussing planetary observations. And this is true. Each observer, in the special theory, synchronizes his own clocks in a way which is equivalent to assuming that the (one-way) velocity of light, as it passes him, is the same in both directions. If a planet's orbit were a material circle, moving round the sun at the proper circular velocity for that distance, it would be possible for an observer to station clocks all round it and to synchronize them, either by light signals (making that assumption) or by actual transport of a chronometer. When he had finished, a stationary observer looking at the whole picture would see that any two of the clocks, separated by the small distance $x$ (as he measures it) would be out of agreement by

$$\frac{vx}{c_r^2}\left(1-\frac{v^2}{c_r^2}\right)^{-\frac{1}{2}} \approx \frac{vx}{c^2},$$

where $v$ is the velocity of the supposed material orbit, i.e. the circular velocity for that radius. Since these discrepancies are all in the same direction, the last clock will differ from the first by their sum; and since these two clocks will be adjacent, the observer moving with them will see the closing error just as undeniably as the fixed one does. In the case of the earth, $v/c \approx 10^{-4}$ and $\Sigma x/c \approx 2\pi \times 499\,\mathrm{s}$; the closing error is thus 313 ms, and even without the comments from the other planet the observer would decide that his procedure was unsuitable. The satisfactory way, he would certainly conclude, is the one which is symmetrical for both planets, and which, if one works it out in detail, produces no closing error for either, namely to allow for the velocity of light in the non-relativistic way. To the first order, this means that the travel times for which radio time signals have to be adjusted, when one intercompares clocks at different observatories, should be evaluated as though the velocity of propagation were not $c$, but $c-v$, where $v$ is the component of the earth's orbital velocity in the direction of propagation, since this is in fact the way an 'impartial' observer, stationary in the solar system, would actually work it out, to the first order in $v/c$. (To the second order, one would of course have to consider also the difference between $c$ and $c_r$, between $a$ and $a_r$, and between $\nu$ and $\nu_r$, i.e. quantities of the order $\psi$, or $v^2/c^2$, but this complication can be omitted.) The travel-time for the earth's diameter is about 42 ms, and the component of $v$ in the appropriate direction is at most $10^{-4}c$; clock comparisons are not yet made with accuracies of 0·004 ms (and would probably be seriously disturbed by vagaries of the ionosphere if they were), but the present accuracy does reach 0·1 ms (p.e.), and the correction is not entirely visionary.* We may add that this way of synchronizing clocks is in agreement with the method now used by the Nautical Almanac for calculating stellar aberration, so far as that has been taken; it is calculated as though the barycentre of the solar system were 'at rest in the ether' and the earth were 'in motion through the ether' at its calculated orbital speed.

We turn, finally, to the problem of giving a rigorous meaning to the radial co-ordinate $r$. As we have already pointed out, this problem is logically of fundamental importance in the relativistic presentation and in the present Euclidean one equally: equation (3) is rigorous itself, and it is essential that $r$ and $\psi$ should have rigorously defined meanings to use in it. The statement that the co-ordinates $(r, \theta, \phi, t)$ are mere arbitrary identification numbers, which is sometimes met with in the literature, is correct if it means that arbitrary analytical substitution formulae can replace any such system by another one, and that the resulting equations in the new system will be correct *if those in the old one were*; but it cannot possibly be regarded as implying that *any* arbitrary identification numbers could be used for $r$, $\theta$, $\phi$ and $t$ in (3), or in (1), as they stand. For any given 'accessible' dimension, such as the radius of a circular orbit in which a particular test particle may actually be moving, there is necessarily one value for $r$, and one only, which can be inserted in (3). It might possibly be suggested that although this is true it does not follow that some experimental method must actually be specifiable which could in principle determine $r$ with an accuracy $m/r$ (so as to distinguish it from $r_1$) or better; but this would be a very unsatisfying viewpoint and is not in fact forced on us.

A method which has sometimes been suggested is simply to measure out the actual geometry of a spherical surface, by measurements confined to that surface; these will naturally give its curvature, and in the particular case of the Schwarzschild solution (1) the radius so found is exactly $r_1$ since tangential measurements are

* Since this was written, successful clock comparisons between England and America have been made, via 'Telstar', with ten times this accuracy; the satellite uses frequencies high enough to penetrate the $F$-layer easily, and ionospheric effects are practically negligible. It is still true, of course, that the apparent movement of any actual planet in 4 $\mu$s is far too small to observe, but the same is not necessarily true of an artificial satellite of the earth.

correct in this system. In the isotropic system, a measuring rod is shortened (73) in the ratio $(1+\psi)^{-2}$ when pointing in the tangential direction (or any other) and the readings will thus all be too large; used uncorrected, they will give $r(1+\psi)^2$, and this of course is in agreement with (2). The observations, however, cannot actually be performed; not only is no such material sphere available, but also it cannot be available, since it would collapse under its own weight unless made of some more or less 'ideal' matter unknown to physicists. It is better not to postulate such magical substances. Without any material sphere, it becomes difficult to guarantee that free measuring rods, whether straight or appropriately curved, would be unaffected by tidal forces and by differential accelerations, and would all be stationary at the same instant ($dt = 0$). We shall therefore employ a different method.

Among the standard ways of determining the solar parallax, one of the best direct ones is to infer the radius of the earth's orbit, in kilometres, from the annual variations in the apparent radial velocities of stars reasonably near the ecliptic. There is no evident reason (apart from the obvious technical difficulties) why the accuracy should not be pushed to any limits desired, and the method does not require calibration of a long terrestrial baseline, nor does it involve any question of simultaneity as between distant stations. We consider therefore a space-ship moving in a circular orbit of radius $r$ (in the sense of our isotropic co-ordinate system), and we require the crew to measure the apparent radial velocity of a star in the ecliptic, and to discover the value of $r$ from the annual variation.

We may suppose the crew to be in possession of any instrument we please, for measuring wavelengths; different ones cannot give different results without enabling the crew to determine their absolute 'motion through the ether'. The instrument we select is an interferometer of the Fabry–Perot type, i.e. two parallel plates of which the first is half reflecting and the second fully reflecting. We suppose the plate separation to be continuously variable down to an effective value (including whatever phase changes are produced) of zero; this zero separation is of course readily identified in white light. If the interferometer is set up in the laboratory, with any monochromatic light source also stationary in the laboratory, and if $N$ fringes are counted as the separation is increased from zero to some positively determined value $l'$ (in 'laboratory' units), then the crew will consider that at any given instant, when the separation is held constant at $l'$, there are between the two mirrors a total of $N$ half-waves going in each direction; they will calculate the wavelength as $\lambda' = 2l'/N$, and they will do this whether they are using a laboratory source or some external one having a relative radial velocity.

Using a laboratory source of a standard wavelength, they can in this way calibrate the length $l'$, but for our purposes this is not in fact necessary. We suppose them to use a sufficiently narrow emission line in the light from a non-variable star exactly in their 'ecliptic'. Let the total fringe-count be $N_1$ when the ship is moving directly towards the star, and $N_2$ when it is moving directly away, half a 'year' later. (It is not, of course, necessary to run the mirror separation down to zero and up again each time; it can be kept constant at $l'$ throughout, and the slow change in $N$ counted directly, as it occurs.) An observer stationary in our Euclidean co-ordinate system cannot disagree with the crew as to the absolute count of fringes from zero separation to the final one, or about the change in the count (from maximum to minimum) in the half period; he will, however, disagree about the interpretation, and in particular will not consider that at any given instant the numbers of 'direct' and 'reflected' waves contained between the two mirrors are equal.

If $\lambda_r$ is this observer's value for the wavelength of the starlight, at the distance $r$ from the origin, and if $l$ is his value for the separation $l'$, he will consider that at any given instant there are $l/\lambda_r$ 'direct' waves between the two mirrors; the reflected waves, when the ship is moving towards the star, will be shortened in the ratio $(c_r - v)/(c_r + v)$ by reflexion at the moving mirror, and there are therefore

$$l(c_r+v)/\lambda_r(c_r-v)$$

of them between the mirrors at this stage. Accordingly

$$\frac{l}{\lambda_r}\left(1+\frac{c_r+v}{c_r-v}\right) = N_1 = \frac{2l'}{\lambda_1'} \tag{79}$$

or

$$\lambda_1' = \frac{l'\lambda_r}{l}\frac{c_r-v}{c_r}. \tag{80}$$

Similarly, half a 'year' later

$$\lambda_2' = \frac{l'\lambda_r}{l}\frac{c_r+v}{c_r} \tag{81}$$

and so

$$\frac{N_1-N_2}{N_1+N_2} = \frac{\lambda_2'-\lambda_1'}{\lambda_2'+\lambda_1'} \equiv \frac{\Delta\lambda'}{\lambda'} = \frac{v}{c_r}. \tag{82}$$

This somewhat elementary discussion has been given in detail because it is essential to avoid all approximations, such as might perhaps be latent in the equation $\Delta\lambda/\lambda = -\Delta\nu/\nu$. It will be noted that the result, down to this point, is independent of the co-ordinate system in use, and holds also for an anisotropic refractive index provided $c_r$ is understood as the *tangential* velocity of light. The value of $l'/l$ has cancelled out, as has $\lambda_r$ (which depends on the star's own radial velocity as well as the possibly unknown atomic transition involved), and no other relativistic term has yet appeared. $\lambda'$ is the immediately observable $\frac{1}{2}(\lambda_1'+\lambda_2')$.

If $\tau$ is the period of revolution, in units of 'true' time, we have $\dot\phi = 2\pi/\tau$. The space-ship's clocks are going slow, both on account of (17) and on account of its velocity; they register the shorter time $\tau'$, where

$$\tau' = \tau\frac{1-\psi}{1+\psi}\left(1-\frac{v^2}{c_r^2}\right)^{\frac{1}{2}}. \tag{83}$$

Accordingly,

$$r = \frac{v}{\dot\phi} = \frac{c_r\tau}{2\pi}\frac{v}{c_r}$$

$$= \frac{c\tau'}{2\pi(1+\psi)^2}\left[1-\left(\frac{\Delta\lambda'}{\lambda'}\right)^2\right]^{-\frac{1}{2}}\frac{\Delta\lambda'}{\lambda'}. \tag{84}$$

The value of $\psi$ can be obtained in terms of $\Delta\lambda'/\lambda'$ by using (61); since $v_c$ in that equation is the circular velocity, we have

$$\left(\frac{\Delta\lambda'}{\lambda'}\right)^2 = \frac{v^2}{c_r^2} = \frac{2\psi}{(1-\psi)^2}. \tag{85}$$

The rigorous value of $r$, in terms of quantities directly measurable by the space-ship's crew, is obtained by eliminating $\psi$ between (84) and (85), and a sufficient approximation to distinguish between $r$ and $r_1$ is obtained by putting

$$\psi \approx \tfrac{1}{2}(\Delta\lambda'/\lambda')^2$$

in (84). The result is

$$r \approx \frac{c\tau'}{2\pi}\frac{\Delta\lambda'}{\lambda'}\left[1-\frac{1}{2}\left(\frac{\Delta\lambda'}{\lambda'}\right)^2\right]. \tag{86}$$

The above analysis could in principle be extended to deal with an 'elliptical' orbit. It cannot, however, be used in the case of the earth itself. For in the classical Newtonian theory, the reduction from the centre of the sun to the barycentre of the earth–sun system involves corrections of approximately $3\times10^{-6}$, while $m/r$ at the earth's distance is about $10^{-8}$; even for Mercury, where the barycentre correction is considerably smaller and $m/r$ is rather larger, the relativity effect is still the smaller of the two. The earth cannot, therefore, be regarded simply as a 'test particle'; it is essential to develop a proper theory of the two-body problem at least. In general relativity, this question was broached in 1937 (Einstein, Infeld & Hoffmann 1937; also Robertson 1937) but it is still being developed; whether any help will emerge from the present treatment cannot, of course, be prophesied, but it is at least conceivable that some additional workers may now be attracted to this field.

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