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AIR FORCE CAMBRIDGE RESEARCH LABORATORIES

L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

A Space—Proper Time Formulation of Relativistic Geometry

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Abstract

Any formulation of the theory of relativity specifies implicitly or explicitly the exactness or inexactness of the temporal and spatial differentials that occur. The Minkowski formulation implicitly assumes the exactness of coordinate (common) time and the inexactness of proper time. In this paper we examine several other possibilities. The assumption of the exactness of proper time and inexactness of common time leads to a space—proper time (SPT) representation of events that (a) yields the customary formal results of the theory including the differential aging prediction of the 'twin paradox,' (b) allows an analog of Fermat's principle to describe both particles and light, and (c) leads to a many—proper time formulation of the relativistic many-body problem essentially equivalent to the Minkowski space formulation. Analogies between this SPT geometry and the geometric approach to thermodynamics, especially as formulated by Carathéodory, suggest the γ function of relativity is an integrating factor with physical meaning for the many-body problem and also provides insight into the concept of virtual photons.

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A Space—Proper Time Formulation of Relativistic Geometry

I. INTRODUCTION

The theory of special relativity has almost from its beginning been characterized by Minkowski space-time geometry. This consists of a pseudo-Euclidean metric constructed from the spatial coordinates and the so-called common or coordinate time. A second type of time, the proper time, has been introduced which coincides with coordinate time for a body at rest in the frame of reference. The proper time is important for it both expresses the timekeeping properties of a moving body and plays a large role in the formulation of 4-vectors. Yet to the best of our knowledge, no one has constructed a space—proper time geometry.

In writing this paper we wish to consider a space—proper time geometry as an alternative relativistic geometry. There are advantages and disadvantages in our approach, both of which we hope to make evident. We feel (and obviously this is our prejudice) that this approach illuminates the meaning of time. A definite disadvantage is the appearance of solipsistic space—proper time diagrams, implying a 'private' world for each particle. A consequence is that such diagrams can describe events with timelike separation only. Recognizing the novelty and strangeness of this approach, we ask the reader to suspend judgment until he has read the paper and considered the potentialities of this geometry.

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The starting point of our investigation is what Costa de Beauregard has called the first law of time. In his book La Notion de Temps (1963) he has pointed out a remarkable parallel between the first law of thermodynamics and the expression for the world line element ds. The first law asserts the equivalence of heat Q and work W,

$$dU = dQ - dW, \tag{1}$$

where U is the internal energy. The expression for the world line element asserts the equivalence of time and space,

$$ds^2 = dr^2 - c^2 dt^2, \tag{2}$$

where dr is the three-dimensional spatial increment, c the velocity of light, and t coordinate time defined by the Einstein clock-synchronization convention. This is also the defining equation of the proper time $d\tau$, for which we write

$$ds^2 = -c^2 d\tau^2. \tag{3}$$

Analogies are admittedly treacherous and rarely if ever allow a one-to-one corresponde ·e. The first law of thermodynamics is a linear relation, whereas the first law of time, Eq. (2), is a quadratic one. The internal energy is a state function of a thermodynamic system and is an exact differential, whereas both the heat and work are path-dependent and are therefore inexact differentials. The first law of thermodynamics thus expresses an exact differential as the difference between two inexact differentials.

Combining Eqs. (2) and (3), we write the proper time as

$$c^2 d\tau^2 = c^2 dt^2 - dr^2. \tag{4}$$

The path element dr is by its very nature path-dependent; this is of course a tautology. The similarity between Eqs. (1) and (2) pointed out by Costa de Beauregard has suggested a need for reexamining the exactness of the differentials appearing in Eq. (2). To pursue the analogy, we wish to consider the proper time as an exact differential while treating the coordinate time as inexact or path-dependent in an appropriate space. This approach draws heavily upon classical thermodynamics, especially as formulated by Carathéodory (1909, 1925).

2. THE LOGICAL BASIS OF SPACE–PROPER TIME GEOMETRY

The Lorentz transformations have the property of leaving the proper time element $d\tau$ of Eq. (4) invariant. Although the path element dr is inexact, it may be written as the sum of the squares of three exact differentials, the usual cartesian coordinates,

$$dr = (dx^2 + dy^2 + dz^2)^{1/2}. \tag{5}$$

This enables us to rewrite Eq. (4) as

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2). \tag{6}$$

This equation lacks physical content until we specify the nature of the time differentials. There are four alternative physical postulates:

(A) dt exact, $d\tau$ inexact.
(B) $d\tau$ exact, dt inexact.
(C) $d\tau$ and dt both exact.
(D) $d\tau$ and dt both inexact.

Minkowski tacitly assumed hypothesis (A) and his results are well known (Lorentz et al, 1958). Hypotheses (C) and (D) will not be considered in this paper, although we shall briefly examine the physical implications of (C) in Sec. 4. We ask the reader to entertain hypothesis (B) as a logical possibility. We hope to show (1) that the results are consistent with the principle of relativity, (2) that the principle of logical economy—Occam's razor—may be invoked in favor of (B), and (3) that observations on macroscopic systems are unlikely to provide a crucial test for distinguishing between (A) and (B).

Equation (6) may be rewritten as

$$c^2 dt^2 = c^2 d\tau^2 + dx^2 + dy^2 + dz^2. \tag{6a}$$

If we restrict ourselves to timelike intervals ($d\tau^2 > 0$) and assume hypothesis (B), dt^2 is then represented by a positive definite sum of squared exact differentials. The quantity cdt may then be interpreted as an arc length in a Euclidean 4-space, just as dr, whose square is a positive definite sum of three squared exact differentials [Eq. (5)], is an arc length in a Euclidean 3-space. It seems to be generally true, though seldom noted, that the natural quantities to use as coordinates in geometric representations of physical processes are those whose differentials are exact. For example, one does not integrate dr directly but rather with respect to dx, dy, and dz. We thus find it reasonable to use x,y,z, and $c\tau$ as the coordinates of the Euclidean 4-space in which cdt is an arc length. This 4-space will be designated as 'space—proper time' (SPT). In the Minkowski representations, which assume postulate (A), the coordinates are taken to be x,y,z and either ict or ct. For convenience and brevity we shall refer to the Minkowski space-time representations as MST.

Before going any further we must admit that SPT necessarily lacks the universal representation capabilities of MST. Since τ is the proper time of an individual particle, SPT is essentially solipsistic. Each SPT must be thought of as a 'private space' belonging to one and only one particle, namely, the one with respect to which the coordinate $c\tau$ is defined. This is not so great a disadvantage as would at first appear. If each particle has its own timekeeping properties unique to the geometry describing the particle, one finds nonsimultaneity of events in different systems a quite natural occurrence. In SPT it is no longer appropriate to call the time defined by Einstein's synchronization convention 'coordinate time' so we shall refer to it as <u>common time</u> or, more explicitly, <u>t time,</u> an expression that has the advantage of being free of any connotations. Its increment $c\Delta t$ will play, as noted, the role of arc length along the SPT world line.

3. THE SPT DIAGRAM AND LORENTZ TRANSFORMATIONS

To make the preceding ideas more concrete let us develop elementary kinematics in SPT. Consider a particle moving with constant velocity along the x axis of an inertial reference frame K. Since the velocity has an x component only, we may suppress the y and z coordinates. Hence, following the ideas of Sec. 2, we use $c\tau$ and x as the coordinates and draw the two-dimensional SPT diagram shown in Figure 1. Since SPT is a flat space with a positive definite metric, the results of Lorentz transformations (for parallel motions representable in the $c\tau$,x plane) become theorems of Euclidean plane geometry. Equation (6<u>a</u>) is simply the Pythagorean theorem for triangle OAB in Figure 1. The points O and A represent two events in the history of the particle moving with velocity v parallel to the x axis, measured in K. The two events are separated by an interval cdt of common time, measured in K. By measurement in K we understand space and common time measurements with Einstein-synchronized clocks and meter sticks <u>at rest</u> in K.

Consider a second inertial frame K' (Figure 2) moving with velocity u with respect to K, with the x and x' axes coincident. The relationship between v, v', and u is derived from SPT principles in Appendix A. The two events represented by O and A in frame K (Figure 1) are represented by O' and A' in frame K' (Figure 2).

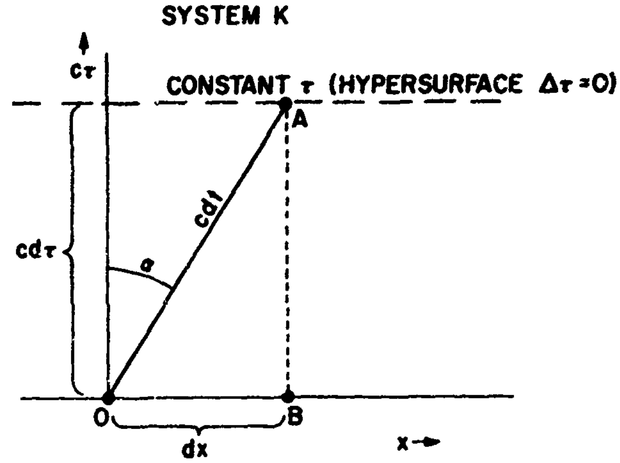

Figure 1. SPT Representation of Trajectory of a Particle Moving Uniformly in the x Direction Relative to the Lorentz System K

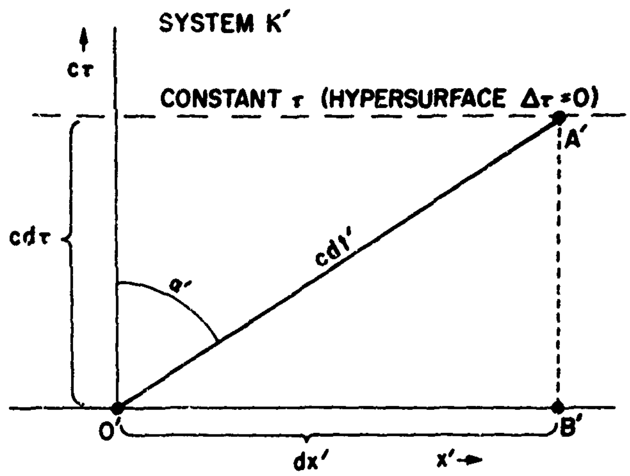

Figure 2. SPT Representation of Trajectory of a Particle Moving Uniformly in the x' Direction Relative to the Lorentz System K' (system K' moves with velocity u with respect to system K)

If we take K and K' as being coincident when the first event occurs and identify O and O' as the origins of K and K', superposition of Figures 1 and 2 shows that A and A' lie on the same line of constant τ. The Lorentz transformations, being those that leave $d\tau$ invariant, are thus transformations of shear on lines (more generally, hyperplanes) of constant τ. It is worth noting that the Galilean transformations of Newtonian physics are also transformations of shear on hyperplanes of constant t, where t in classical physics is an absolute time parameter. Since classical mechanics contains no universal constant c, however, the 'space—absolute time' (SAT) in which such transformations occur is dimensionally inhomogeneous and therefore not directly comparable to the homogeneous 4-spaces of relativity theory. The absence of a universal constant makes it impossible to assign a natural dimension to the arc length of the classical world line but does not preclude formation of mechanical variational principles in SAT.

Returning to Figures 1 and 2 we note that the quantity τ, plotted as $c\tau$ on the ordinate axis, is in all cases the proper time of the particular particle represented. [An abstract 'proper time' independent of the particle has no physical meaning. The proper time τ is the time shown by a clock co-moving with the particle. In general, this clock is not permanently at rest in any inertial system. This is the origin of the solipsistic nature of SPT.] The SPT world line (OA in Figure 1) of any macroscopic particle always exists in a fixed relationship to its SPT axes, since at every 'age' the particle occupies a definite spatial location. Thus, a one-to-one correspondence exists between a sequence of contiguous representation points in SPT and the dynamic history of the particle. Further proof of the physical 'existence' of space—proper time is not needed nor can it be given. We cannot plot world lines for more than one particle in one SPT diagram. Nor can we conceive of such a diagram in the absence of any particle. In this sense SPT is concrete.

The shear-related representations of SPT geometrically express the principle of relativity. Observers in different inertial frames who plot the first of two events associated with a given particle at a common SPT origin will plot the second at various loci on the same constant τ hyperplane. Each locus refers to a different Lorentz frame, but all loci are connected through shear transformations. Thus, the SPT representation of events, like that of MST, is not unique or absolute but is physically indeterminate within a Lorentz transformation in conformity with the relativity principle. The difference is that in MST the world lines remain invariant while the axes move under a Lorentz transformation (rotate if ict is the temporal coordinate, scissor together if ct is). In SPT the axes remain orthogonal for all frames while the world line moves by shearing. Classical SAT has features in common with both. It resembles SPT in the nature of its transformations but resembles MST in possessing the capability of describing simultaneously any number of particles. The invariance of the SPT axes under Lorentz transformations prevents portrayal of relative motion between two inertial frames. The relative motion between the frames must be inferred from the difference in slopes of the two world lines plotted in the two appropriate SPT diagrams (see Figures 1 and 2).

It must be noted that SPT cannot describe events of spacelike separation simply—again owing to the solipsistic nature of the representation. It can represent simply only events of timelike separation on the world line of a single particle. Spacelike separations must of necessity refer to two or more particles. A striking feature of SPT diagrams is thus the absence of the 'elsewhere' beyond the light cone in MST. Superficially, the situation resembles the one that might obtain were the forward and backward light cones associated with a particular event in MST squeezed together into a single hyperplane ($\Delta\tau = 0$). This squeezing together would exclude the 'elsewhere,' eliminate the distinction between advanced and retarded interactions, and exaggerate the interior portions of the light cones (past and future).

Despite the apparently nonphysical nature of SPT, we verify in Appendix A that SPT yields the familiar formal results of elementary relativity theory. This is not surprising since the SPT representation satisfies Einstein's original postulates of relativity and the constancy of light velocity.

4. THE INEXACTNESS OF dt AND CARATHÉODORY'S PRINCIPLE

A cursory look at Sec. 3 and Appendix A might suggest that SPT is merely a different geometric representation of physical facts long familiar from MST analysis. We feel that examination of the SPT description of the propagation of light clarifies the physical meaning of several ideas underlying relativity. Also, the thermodynamic analogy has led, we believe, to a physical interpretation of the γ factor. This interpretation may prove helpful in formulating the relativistic many-body problem.

Consider the propagation of light. In SPT as in MST, the path of a photon is specified by $\Delta\tau = 0$. For photon motion in the x direction, differentials are integrated to give finite increments $\Delta x = c\Delta t$ and from Eq. (6) we have

$$c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 = 0.$$

In SPT, therefore, a photon during its entire life is confined to a hyperplane of constant τ. Its one-way propagation, which in classical terms may be pictured as occurring on an expanding spherical light shell, is unambiguously described by increasing values of the parameter t. As soon as we encounter any deviation from one-way motion however—as in the case of reflection by a mirror—a new consideration enters.

Our SPT analysis has thus far been based solely on an assumption about $d\tau$, namely, that it is an exact differential. We have not as yet made explicit use of the other half of postulate (B), to wit, that dt is inexact. Hence, the development to this point is compatible with both postulates (B) and (C).

Before examining postulates (B) and (C) let us first rewrite Eq. (6):

$$d\tau^2 = dt^2\left[1 - (dx/dt)^2/c^2\right], \tag{7}$$

or

$$d\tau = dt(1 - v^2/c^2)^{1/2},$$

where v is the 3-velocity in the frame considered. If we define γ as $(1 - v^2/c^2)^{-1/2}$, we may write

$$d\tau = dt/\gamma \, . \tag{7a}$$

Since v equals c for a photon, γ is infinite and $d\tau$ is zero, regardless of the value or exactness of dt. Were postulate (C) correct—both $d\tau$ and dt exact—the photon could propagate outward from the origin as an expanding light shell, reverse its direction at any moment, and return on a contracting light shell, remaining always on the hyperplane $\Delta\tau = 0$ in SPT. On the outward journey its progress would be represented in a two-dimensional diagram like Figure 1 by a lengthening vector of magnitude $c\Delta t$ parallel to the x axis. At the turn-around time, Δt_1 after emission, the photon SPT world line would attain a maximum length and then begin to shorten. The resultant vector would eventually shorten to zero length, implying the photon's return to its source at the same t time as its virtual emission. This is of course the meaning of exactness of dt: the photon completes a round trip without any change in t, which implies $\oint dt = 0$. Breaking the journey into many short t-time increments Δt_i shows that the process just described is equivalent to assuming vectorial additivity of the SPT world line elements $c\Delta t_i$. The exactness of dt, the microscopic reversibility of t time, and the vectorial additivity of t-time increments in SPT are thus three equivalent ways of describing particle behavior conforming to postulate (C).

Can we ascribe physical reality to such behavior? Nothing in our experience with completed or observable processes corresponds to it. But for quantum virtual processes described by time-reversible wave equations and occurring prior to quantum mechanical measurement, the system remains in a quantum pure state and is therefore thermodynamically isolated. Thus, the condition of macroscopically reversible and isentropic behavior for t time is met, and postulate (C) is probably the physically relevant one. Examples of such a process are found in von Weizsäcker's (1934) virtual photon model of the nuclear field and in the quantum electrodynamic description of the Coulomb field. Such photons are presumably described by postulate (C). For the rest of this paper we shall restrict ourselves to observable macroscopic processes, for which we reassert postulate (B).

The inexactness of dt implies that successive increments of $c|\Delta t_i|$, represented by segments of arc length in SPT, are <u>scalar additive</u>. Hence, the irreversibility of t time is implicit in the very nature of the SPT description. Let us return to the reflection of a photon by a mirror. Suppose that at time $\Delta t_1 = \Sigma|\Delta t_i|$ after emission the photon encounters a mirror and is reflected back toward its source. Is the reflection event itself a physically irreversible process? One might think not, because the photon appears to remain in a phase-connected pure state describable by t-time reversible equations. Moreover, no 'observation' need have been made on the system in the strict Copenhagen sense. Nevertheless, something irreversible has taken place at the mirror as is shown by the occurrence of radiation reaction. After mirror recoil, propagation of the photon can no longer be virtual—the photon has 'lost the option of changing its mind.' Indeed, it can no longer be the same photon since it must have a slightly lower frequency. This second photon, like the first, must be described by a steadily increasing Δt. The description of such two-way propagation is easy in MST but requires caution and is susceptible to ready misinterpretation in SPT.

Since SPT diagrams are different for different particles, the two photons cannot be represented in one diagram. Each photon is represented in its own SPT on its own hyperplane of constant τ. Otherwise, $\Delta\tau = 0$ would imply that after reflection the photon returns to the observer at the same proper time it was emitted. Actually this would be the same as attempting to crowd three different proper times—those of the two photons and that of the observer—into one SPT diagram. To treat two or more particles, separate SPT diagrams must be introduced. The problem of collective description is then one of correlating the single particle SPTs.

If we assume postulate (B), Eq. (7a) becomes highly suggestive. It is at this point we wish to introduce our 'thermodynamic' argument. Appendix B contains a brief summary of the Carathéodory formalism. By dividing the inexact differential dt by γ we convert it into an exact differential $d\tau$. Therefore γ is an integrating factor of dt in SPT. Now we know that a Pfaffian differential expression with two variables always possesses an integrating factor. The Pfaffian form for t time would then be dt = 0. In thermodynamics the inexact differential of heat is dQ. The condition for adiabaticity is that dQ equal zero. We conclude that the virtual photon plays the same role in our SPT geometry as adiabaticity does in thermodynamics. Both concepts—quantum virtual processes and adiabatic processes—also share the quality of being idealized system 'motions' that are strictly unobservable. It should, however, be noted that adiabaticity, although a necessary condition for thermodynamic reversible behavior, is not sufficient in itself. An adiabatic process is not necessarily isentropic, as witness a Joule expansion. Consequently, the entropy differential dS is not always equal to dQ/T, and so we write the inequality:

$$dS \geq dQ/T\,. \tag{8}$$

In this sense we do not have an exact correspondence between zero dt and zero dQ, for the virtual photon must be reversible. Moreover, Eq. (7a) is an equality, not an inequality, and we can always integrate it for real photons and material particles as well as for virtual photons. The SPT formalism is thus even simpler than that of thermodynamics.

From the SPT analysis we have arrived at the following ideas, which are admittedly highly speculative but nonetheless suggestive. In thermodynamics the integrating factor is more than a formal mathematical device. The temperature has profound physical significance as a statistical measure of the energy of a system. It is used in the Clausius formulation of the second law: heat cannot flow from a body at a lower temperature to one at a higher temperature without mechanical work having been done. We therefore ask the more general question: If an integrating factor appears in the analysis of a physical problem, does it have physical significance? Since we interpret the γ function as an integrating factor, we seek such an interpretation. We should note here that whether we accept postulate (A) and MST, or accept postulate (B) and SPT, Eq. (7a) expresses a relation between an exact and an inexact differential. Any physical interpretation of γ is therefore not restricted to the SPT formalism.

In attempting to ascribe a physical meaning to the γ function we look for clues in the meaning of temperature. Temperature is a statistical measure of the energy of a system—to speak of the temperature of one electron is meaningless. The γ function can be written for a single particle and so is not statistical. Involving velocity as it does, γ is therefore related to energy, and so is characteristic of the state of a particle. For any given particle, γ has a minimum value of unity (particle stationary in the frame) and a maximum value of infinity for a photon. The reciprocal of γ therefore lies in the range between zero and unity. To attempt to characterize a system of N particles let us define Γ^{-1} as

$$\Gamma^{-1} = \sum_{i=1}^{N} \gamma_i^{-1}, \tag{9}$$

where γ_i is the value of γ of the ith particle. The maximum value of Γ^{-1} is N and the minimum value is zero. Can we use the change in Γ^{-1} as a measure of change in the system? To answer this question let us examine several simple processes.

In pair-creation a gamma ray having energy greater than 1 MeV disappears and an electron and positron are created. The original gamma ray had a γ^{-1} equal to zero. Each of the pair created has a γ^{-1} equal to η, where η lies between zero and unity. The change in Γ^{-1} of the system is therefore $+2\eta$. For the reverse process of annihilation the change is -2η.

As a second process consider the absorption of a photon by a stationary hydrogen atom in the ground state. Before absorption the proton and electron each have a γ^{-1} of unity and that of the photon is zero. The absorption of the photon leads to motion of the atom, whether or not there is ionization. Therefore the γ^{-1}s of the proton and electron both decrease to η_p and η_e. For the process the net change in Γ^{-1} is $2-(\eta_p + \eta_e)$.

These examples suggest the formulation of a general principle. We propose the principle in a tentative fashion. Even if it is valid (as we believe), its usefulness remains to be shown. The principle may be stated as follows: If the number of particles of a system decreases, the quantity Γ^{-1} of the system also decreases; if the number increases, the quantity Γ^{-1} also increases. Note that this is not related to the energy or momentum content of the compound system since each γ_i is a function of velocity alone, not of mass or energy. We feel the notion can be applied to compound particles (as witness the hydrogen atom) and perhaps even to continuous media. This is of course highly speculative. In our treatment we have assumed that the system under discussion is isolated. If, during the process considered, the state of motion of the system were changed, the above arguments would be invalid.

5. KINEMATICS OF NONUNIFORM MOTION AND THE POSTULATES OF RELATIVITY

A preliminary discussion of the logical structure of the theory is helpful in the development of the SPT kinematics of nonuniform motion. Einstein's two original postulates—the principle of relativity and the constancy of light velocity—are compatible with physically different representations of events such as SPT and MST. It is therefore quite evident that

a) As physics, the two-postulate system is incomplete.

b) Any particular geometric representation of physical events such as MST contains logical implications that go beyond those of Einstein's two original postulates.

(For an interesting discussion of the Einstein postulates see Terletskii, 1968.)

In the past the adequacy of special relativity to deal with accelerated motions has been the subject of debate. Present considerations suggest that if the special theory is based solely on the incomplete system specified by Einstein's two postulates, then it cannot deal with accelerated motions. If, however, the theory is augmented by a postulate specifying the nature of the time differentials, then it can deal with such motions. The MST representation implicitly introduces postulate (A). The resulting three-postulate version of special relativity is competent to describe world lines of any physically admissible shape in flat space. This is why the MST description resolves the twin paradox, even though logical purists, who admit the two original postulates only, maintain that the so-defined special theory cannot be applied to the question.

Einstein himself appears to have been of both minds. At first he gave the conventional differential-aging prediction for the twins, but ultimately he retired to the safe but sterile position that flat-space analysis is inapplicable to curved world lines. This view, strictly enforced, would deny respectability to Einsteinian pre-gravitational mechanics and the Dirac equation for the electron. Einstein apparently did not realize that his original flat-space analysis was postulationally underdetermined for all applications beyond the kinematics of uniform one-way motion.

It should be added that our emphasis on the physical need for a third postulate concerning the nature of the time differentials does not by any means imply a belief in the sufficiency of three postulates for a physically complete theory. Rather, we must agree with Synge (1965) that a truly complete enumeration of postulates and definitions probably lies beyond present capabilities and settle in practice for a heuristic theory that emphasizes the principal postulational elements.

Let us now consider the twin paradox by using SPT. We assign an individual SPT to each twin and make no hypothesis concerning any geometric relationship between different SPTs. Consider the traveling twin as No. 1. At proper time $\tau_1 = 0$ after instantaneous acceleration, he departs at speed v and travels a spatial distance Δx along a track such as OA in Figure 1. On this part of the journey he ages an amount $\Delta\tau_1$, given by Eq. (7a) as $\Delta\tau_1 = \Delta t/\gamma$. On the return journey at speed v from point A in Figure 1 to $x = 0$ along a line of reverse slope (not shown), he ages an equal amount, so that his total aging is $2\Delta\tau_1$. Since postulate (B) is assumed for SPT, dt is inexact and all time increments $|\Delta t_i|$ are scalar additive along the traveler's track. The total elapsed t time is therefore twice Δt, the value for the outward journey; and the relation between $\Delta\tau_1$ and Δt given above applies to the whole journey.

The total elapsed common time, $2\Delta t$, is the amount by which the stay-at-home twin, No. 2, ages during his brother's journey. We can see this by the following reasoning. In his own SPT, with proper time axis τ_2, twin No. 2's world line is a straight vertical track at $x = 0$. Since $\Delta x = \Delta y = \Delta z = 0$ in this space, it follows from Eq. (2) that $2\Delta\tau_2 = 2\Delta t$, the total elapsed time during the traveler's absence. Hence

$$2\Delta\tau_2 = 2\Delta t = \gamma 2\Delta\tau_1 . \tag{10}$$

Therefore, in SPT based on postulate (B), just as in MST based on postulate (A), the stay-at-home twin ages by a factor γ more than the traveler. The only assumption made in the above derivation is that the same t time is measured along the arc lengths in SPT_1 and SPT_2. This is trivially true, since the same physical set of Einstein-synchronized clocks is the referent in both cases, namely, the clocks at rest in the inertial system in which both twins are originally at rest.

Just as in the case of mirror reflection, total confusion results from any attempt to crowd both twins into the same SPT. If both twins were to use the τ axis in Figure 1 to register a hypothetical common proper time, the elapsed τ time between departure and return would be the same for both twins. This is obvious from the fact that the traveler's world line would intersect the τ axis at the two events. This is similar to the error in photon description whereby the photon apparently returned at the same time as it was emitted.

It thus appears that postulate (B) and the SPT representation of events are not incompatible with a satisfactory kinematic description, provided separate SPTs are assigned to each particle. But this is not a satisfactory many-body description. For a more adequate many-body description we must seek a geometric correlation between the individual SPTs that will at the same time clarify the sense in which t time may be said to be conventional.

6. SPT AND THE COLLECTIVIZING CONVENTION

It is apparent from the nature of SPT that any treatment of the relativistic many-body problem based on it must at the most fundamental level be a many—proper time theory rather than a shared time theory. In principle we might consider the various 4-spaces, each space associated with an individual particle, as completely uncorrelated, since the total information content of these spaces is equivalent to that of any collective space description. Such independence would not allow a simple formulation of particle interactions and equations of motion.

If, in the example of the twins just given, we try to synthesize a collective space representation by superposing the two SPTs in some fixed geometric relationship, we would have to represent the event of the traveler's return by two distinct points, with the departure represented by one. To see this, let the event of departure occur at $t = 0$. For the traveler, his return occurs at $t_R = 2\Delta\tau_1$; but for the stay-at-home twin, it occurs at $t_R = \gamma 2\Delta\tau_1$. To obtain a one-to-one correspondence between physical events and mathematical representation points, we must therefore find a way to continually adjust the different SPTs. This is equivalent to requiring a nonstatic relationship between the SPTs of the individual particles. A convention must be found that will allow the individual SPTs to be moved with respect to each other concomitantly with the passage of t time. This will enable us to satisfy the requirement of one-to-one correspondence. With no unique convention of this kind, simplicity, linearity, and familiarity recommend the one that yields the Minkowskian description. The conventions to be discussed are of some mathematical interest, illustrating as they do the interconvertibility of exact and inexact differentials.

The simplest system to consider is that of a photon having velocity c and a point particle at rest as seen by an observer at rest in an inertial frame K. Let this be the laboratory frame. Each particle is assigned its own private $(SPT)_i$.

For the photon, $i = 1$; for the particle, $i = 2$. The spatial axes of $(SPT)_1$ and $(SPT)_2$ measure the position (x_i, y_i, z_i) of the appropriate particle in the K frame, and the ordinate axis measures the proper time of the appropriate particle. The two τ_is are distinct for the two particles and are related through Eq. (7a) to the t time t_o measured by the observer,

$$d\tau_i = dt_o/\gamma_i .$$

Since γ_1 is infinite and γ_2 is unity, we have

$$d\tau_1 = 0; \quad d\tau_2 = dt_o .$$

Each SPT, although referring to the same 3-space, thus measures a different proper time.

To obtain a collective description of the photon and particle we superpose $(SPT)_1$ and $(SPT)_2$, assigning a common spatial origin and keeping the spatial axes parallel. Let P_i, a point on the particle world line fixed in the SPT_i diagram, represent the present position ($t = 0$) of the ith particle. We adjust SPT_1 and SPT_2 so that P_1 and P_2 are brought to a common height along their superposed $c\tau_i$ axes. In the next instant of t time each particle describes the arc length $c\delta t$ along its SPT world line. For the photon, τ remains unchanged and the SPT_1 world line increment is perpendicular to the photon $c\tau_1$ axis. For the particle at rest in the laboratory frame, the world line increment is parallel to the particle $c\tau_2$ axis and normal to the spatial axes. These represent the two extreme cases. A particle in K with velocity v less than c but greater than zero would have a world line increment making an acute angle α with its $c\tau$ axis. To bring the two $(P_i + c\delta t)$ points to a new common altitude, the photon SPT must be moved a distance $c\delta t$ along the common direction of the proper time axes while the photon itself moves a distance $c\delta t$ parallel to the spatial axis. The photon would then describe a path at 45^{o} to the vertical axis of the collective space S_M in which both particles are located. In this collective space, which is the real coordinate Minkowski space corresponding to K, the vertical axis is now labeled ct, and the loci of the P_i in S_M describe the corresponding MST world lines. Such a collectivization is termed linear or Minkowskian.

To repeat, this underlying or common space representation in which the individual P_is trace out their Minkowski world line loci is achieved by moving the points P_i (present positions of the particle in SPT_i) a vertical distance $c\delta t$ as the particle progresses a distance $c\delta t$ along its world line fixed in SPT_i. Concomitant with this upward motion of P_i by an amount $c\delta t$ there is an upward displacement of the entire SPT_i system by an amount $c\delta t(1-\cos\alpha) = c\delta t(1 - \gamma_i^{-1})$, as may be verified from the geometry of Figure 1.

The generalization to N point particles moving with generally different varying velocities v_i is obvious. For each particle the relation $d\tau_i = dt_o/\gamma_i$ holds, where dt_o is again defined as the common time as measured by a clock at rest in K. The particular K frame to which all the SPTs are referred is a Lorentz frame, with spatial and t-time quantities operationally defined as in Einstein's theory. Each proper time axis $c\tau_i$ is calibrated by the rule that when the ith particle is at rest in K, any increment $\Delta\tau_i$ is identical with the corresponding Δt measured by a clock at rest in K. Note that we have not restricted the particles to uniformly translating motion. Accelerations are allowed.

We have thus shown the existence of a simple convention whereby the real coordinate Minkowski space representation of events can be synthesized through superposition of SPT single particle representations. The 'slippage' of one SPT frame past another, introduced by our collectivizing convention, acts to reverse the nature of t and τ differentials. The convention in effect makes dt exact and $d\tau$ inexact. Thus, in S_M, postulate (A) applies. The particle world line in SPT is not the same as that in the collective MST but the two are related by elementary geometry. By offering a clearer appreciation of the conventional nature of t time, the SPT representation provides something more than a rederivation of old results, quite apart from the advantage of a true Euclidean metric. Advantages of perhaps equal importance are the new geometric interpretation of proper time and the significance attached to γ as an integrating factor.

7. DISCUSSION

A significant result of SPT analysis is a unification of photon and particle descriptions. The inexactness of dt and the use of dt as an arc length parameter in SPT allow us to describe either particle or photon trajectories by the geodesic equation

$$\underset{(\min)}{\delta} \int_{SPT} dt = 0 . \quad \text{(particles or photons)} \tag{11}$$

The extremum indicated here is a minimum for the actual path between fixed endpoints in SPT relative to nearby alternative paths. Since Eq. (11) is formally identical with Fermat's principle for light paths, the unification of photon and particle descriptions is evident. Such unification suggests that there may be a more fundamental physical significance in the SPT description than in the MST description, although the latter will always retain its practical usefulness.

In the MST description, the exactness of dt, the inexactness of $d\tau$, and the use of $d\tau$ as an arc length parameter of the MST world line, lead to the more familiar geodesic equation

$$\underset{(\max)}{\delta} \int_{MST} d\tau = 0 . \quad \text{(particles only)} \tag{12}$$

The extremum here is a relative maximum. It lacks the generality of Eq. (11) since it does not describe the photon paths. The customary MST representation thus creates a distinction between photons and material particles contrary to much of the spirit of modern physics. This distinction, which has been built into the most widely accepted forms of both the special and the general theories of relativity, may prove to be an Achilles heel.

In summary the postulational situation appears to be as follows. Under our collectivizing convention, straight line geodesics in SPT transform into straight line geodesics in MST. Hence, observation of uniform motion provides no way of distinguishing postulate (A) from postulate (B).

Our discussion of the twin paradox similarly indicates that no distinction arises in the observation of accelerated motions, at least in the nonquantum macroscopic domain of sharply defined world lines. Since Eq. (11) describes both photons and material particles, whereas Eq. (12) describes only the latter, logical economy would in the absence of contrary observational evidence appear to favor postulate (B) as more fundamental. It must, however, be remembered that variational principles, like covariance or relativity principles, are in themselves no panacea for the physicist. The physics lies not only in the principles but also in the assumptions defining the time-space descriptive matrix in which the principles are applied.

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Appendix A

Elementary Kinematics in SPT

We wish to show that we can obtain the formal results of elementary relativity theory from the SPT representation. In Figure 1 a particle moving with uniform velocity v with respect to K describes the world line $\overline{OA}$. The uniformity of motion allows us to replace differentials by finite increments, and we write

$$\sin \alpha = \overline{OB}/\overline{OA} = \Delta x / c\Delta t = v/c. \tag{A1}$$

Moreover,

$$\begin{aligned} \cos \alpha &= \overline{AB}/\overline{OA} = \Delta \tau / \Delta t \\ &= (1 - \sin^2 \alpha)^{1/2} \\ &= (1 - v^2/c^2)^{1/2} \end{aligned} \tag{A2}$$

If we consider the same particle described relative to inertial system K' by the SPT world line $\overline{O'A'}$ of Figure 2, we have

$$\sin \alpha' = \overline{O'B'}/\overline{O'A'} = \Delta x'/c\Delta t' = v'/c, \tag{A3}$$

$$\cos \alpha' = \overline{A'B'}/\overline{O'A'} = \Delta \tau / \Delta t', \tag{A4}$$

where v' is the velocity of the particle with respect to K'. Equations (A2) and (A4) express time dilatation. From these four equations we can derive the elementary Lorentz transformations:

$$\Delta x' = \gamma(\Delta x - u\Delta t), \tag{A5}$$

$$\Delta t' = \gamma(\Delta t - u\Delta x/c^2), \tag{A6}$$

where

$$\gamma = (1 - u^2/c^2)^{-1/2} \tag{A7}$$

and

$$u = (v - v')/(1 - vv'/c^2). \tag{A8}$$

We interpret u as the velocity of K' with respect to K.

Proof

From Eqs. (A3) and (A4),

$$\Delta x' = v'\Delta t' = v'\Delta\tau/(1 - v'^2/c^2)^{1/2}.$$

From Eqs. (A2) and (A8),

$$\Delta x' = v'\Delta t\left(\frac{1 - v^2/c^2}{1 - v'^2/c^2}\right)^{1/2} = \gamma(v - u)\Delta t.$$

Using Eq. (A1) then yields Eq. (A5). Similarly, from Eqs. (A2), (A4), and (A8),

$$\Delta t' = \Delta\tau/(1 - v'^2/c^2)^{1/2} = \Delta t\left[(1 - v^2/c^2)/(1 - v'^2/c^2)\right]^{1/2}$$

$$= \gamma(1 - uv/c^2)\Delta t,$$

which with Eq. (A1) yields Eq. (A6).

Equations (A5) and (A6) (supplemented by the relations $\Delta y' = \Delta y$; $\Delta z' = \Delta z$) are the equations of the Lorentz transformation, and Eq. (A8) expresses the Einstein velocity-composition law. From the Euclidean geometry of SPT we have therefore deduced the simplest kinematic relationships forming the core of special relativity theory. This geometry is based on the invariance of the 4-dimensional line element of Eq. (2) and postulate (B).

Appendix B

A Précis of Carathéodory's Principle and Thermodynamics

Born (1922) and Buchdahl (1949) have each written quite complete expositions of Carathéodory's ideas (1909, 1925). A succinct statement of Carathéodory's principle is Buchdahl's: "In the neighborhood of any arbitrary initial state J_o of a physical system there exist neighboring states J which are not accessible from J_o along adiabatic paths." This is the physical statement dealing with the solutions of a Pfaffian differential expression.

Consider a Pfaffian expression of the form

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)\,dz = 0. \qquad \text{(B1)}$$

This equation is integrable if and only if in the neighborhood of any arbitrary point G_o there are points G that are inaccessible from G_o along solution curves of Eq. (B1). This is equivalent to stating that Eq. (B1) is integrable, if there exist functions $\lambda(x,y,z)$ and $F(x,y,z)$ such that

$$Pdx + Qdy + Rdz \equiv \lambda dF. \qquad \text{(B2)}$$

In thermodynamics the first law yields an equation of the type of Eq. (B1). The heat absorbed by a system undergoing a quasistatic adiabatic process is zero, and is written as

$$dQ = \left(\frac{\partial U}{\partial v} + p\right) dv + \frac{\partial U}{\partial \theta} d\theta = 0, \qquad \text{(B3)}$$

where U is internal energy, p pressure, v volume, and θ temperature. In general, dQ is not integrable. From Carathéodory's principle, however, we can show that Eq. (B3) implies the existence of a state function S such that dS = 0 for this quasi-static adiabatic process and an integrating factor T such that

$$dS = dQ/T. \tag{B4}$$

This equation expresses the relation between the exact differential of entropy and the inexact differential of heat.

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13. ABSTRACT

Any formulation of the theory of relativity specifies implicitly or explicitly the exactness or inexactness of the temporal and spatial differentials that occur. The Minkowski formulation implicitly assumes the exactness of coordinate (common) time and the inexactness of proper time. In this paper we examine several other possibilities. The assumption of the exactness of proper time and inexactness of common time leads to a space—proper time (SPT) representation of events that (a) yields the customary formal results of the theory including the differential aging prediction of the 'twin paradox,' (b) allows an analog of Fermat's principle to describe both particles and light, and (c) leads to a many—proper time formulation of the relativistic many-body problem essentially equivalent to the Minkowski space formulation. Analogies between the SPT geometry and the geometric approach to thermodynamics, especially as formulated by Carathéodory, suggest the γ function of relativity is an integrating factor with physical meaning for the many-body problem and also provides insight into the concept of virtual photons.

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