Minkowski versus Euclidean 4-vectors

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Abstract

Minkowski 4-vectors are written in Euclidean form. It is shown that in this way the physical meaning of their components can be made more intuitive and directly associated with geometric properties in Euclidean space-time.

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1 Displacement

If the Minkowski 4-vector components for displacement in 4D space-time

$$ds^2 = d(ct)^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

are alternatively written in Euclidean form (see also [1]) they read:

$$d(ct)^2 = ds^2 + dx^2 + dy^2 + dz^2 = d(c\tau)^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

Equations 1 and 2 both contain the same information but the essential difference is that the roles of the variables have changed. In Eq. 1, ds^2 is the Lorentz invariant. In Eq. 2, dct^2 is the Lorentz invariant. The mathematics of this switch remain consistent with special relativity. Equation 2 shows that the speed in the Euclidean time dimension, which is now formed by the proper time τ is:

$$v_\tau^2 = c^2 - v_{space}^2 \tag{3}$$

The displacement in the proper time dimension for a moving object in an interval dt (according to an observer at rest) equals:

$$\begin{aligned} v_\tau^2 dt^2 &= c^2dt^2 - v_x^2dt^2 - v_y^2dt^2 - v_z^2dt^2 \\ &= c^2dt^2 - dx^2 - dy^2 - dz^2 \\ &= ds^2 \end{aligned} \tag{4}$$

So ds is now no longer the invariant Minkowski displacement but the displacement in the proper time dimension. The factor dct that played an equivalent role in the Minkowski 4-vector has become the invariant 4D displacement in Euclidean space-time. This is visualized in Fig. 1. Here the spatial displacement $\sqrt{dx^2 + dy^2 + dz^2}$ is written as a single variable dA. The 'Minkowski triangle' (left) then is:

$$ds^2 = dct^2 - dA^2 \tag{5}$$

and the 'Euclidean triangle' (right) is:

$$dct^2 = ds^2 + dA^2 \tag{6}$$

The dotted lines in Fig. 1 represent the values for dA and dct that would result from a Lorentz transformation.

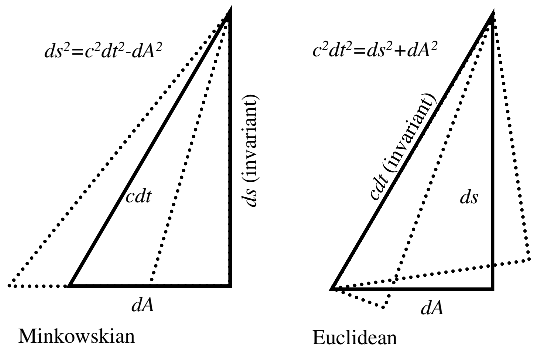

Figure 1: Geometric visualization of Minkowski and Euclidean displacement 4-vectors. Spatial displacement is represented as dA.

2 Velocity

We will henceforth use, conform [1], the notation $\chi = ds/dt = cd\tau/dt$ for the speed in the proper time dimension and v for spatial speed. An attractive benefit in the Euclidean components of the 4-vector for velocity is that they are now the regular derivatives of the displacement 4-vector components with respect to t:

$$\left(\frac{d(ct)}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 + \left(\frac{dA}{dt}\right)^2 \tag{7}$$

or

$$c^2 = \chi^2 + v^2 \tag{8}$$

In Fig. 2 this is again illustrated.

The corresponding Minkowski 4-vector components for velocity are derived by multiplying the time derivative of the Minkowski displacement 4-vector with the factor $\gamma = 1/\sqrt{1 - v^2/c^2}$,

$$c^2 = \gamma^2(c^2 - v^2) \tag{9}$$

which is the same as taking the derivative with respect to the proper time $\tau = t/\gamma$ instead of t. Without this factor the 4-vector does not yield a Lorentz invariant value. Figure 3 shows the background of this multiplication and here it shows that, from an Euclidean perspective, the Minkowski 4-vector components seem to have a pure mathematical function only. The components of the Euclidean 4-vector on the other hand have an intuitive physical meaning: they represent the actual speeds in Euclidean space and proper time. The vector-sum of these has universal magnitude c and *rotates* in 4D if the spatial velocity accelerates in 3D.

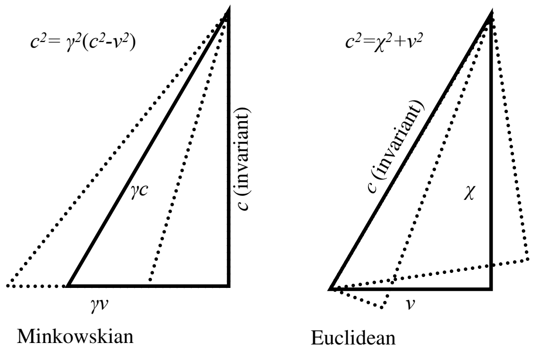

Figure 2: Geometric visualization of Minkowski and Euclidean velocity 4-vectors.

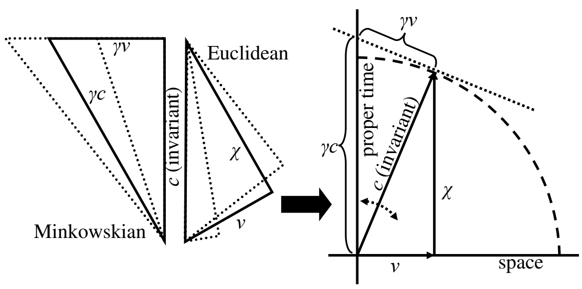

Figure 3: Geometry of Minkowski and Euclidean velocity 4-vectors.

3 Acceleration

Figure 4 shows the geometry for the components of, respectively, the Minkowski 4-vector for acceleration $\alpha_{\mathbf{M}}$ and the Euclidean one, $\alpha_{\mathbf{E}}$. Here, a different geometric representation is chosen for the factors γc and γv to remain consistent with the direction of spatial velocity vector $\mathbf{v}$ (note that the scalar values of these factors have been used in the previous examples). The components of the Euclidean acceleration 4-vector are

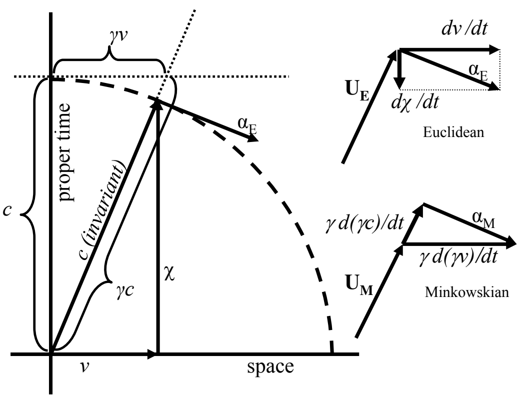

Figure 4: Geometry of Minkowski and Euclidean acceleration 4-vectors.

$$\alpha_{E_\mu} = \frac{dU_{E_\mu}}{dt} = (d\chi/dt, d\mathbf{v}/dt) \qquad (10)$$

Any acceleration in 3D space corresponds to a rotation in SO(4) of the Euclidean 4D velocity vector $\mathbf{U_E}$ with invariant magnitude c, implying that the acceleration 4-vector must always be orthogonal to it, or $\alpha_{\mathbf{E}} \cdot \mathbf{U_E} = 0$. This is in agreement with the Minkowski acceleration 4-vector $\alpha_{\mathbf{M}}$ with components

$$\begin{aligned} \alpha_{M_\nu} &= dU_{M_\nu}/d\tau \\ &= [\gamma d(\gamma c)/dt, \gamma d(\gamma \mathbf{v})/dt] \\ &= \gamma(c d\gamma/dt, \mathbf{v} d\gamma/dt + \gamma d\mathbf{v}/dt) \quad (11) \end{aligned}$$

that is also orthogonal to the Minkowski velocity 4-vector. The components for the Minkowski acceleration 4-vector have their origin in the change of the line elements γc and γv as is shown in the Figure. As these line elements were shown to have a mathematical function only in Euclidean space-time, the derived acceleration components will either. In the Euclidean 4-vector on the other hand, dv/dt and $d\chi/dt$ form the orthogonal vector components and these maintain their intuitive physical interpretation, while the magnitude of $\alpha_{\mathbf{E}}$ is invariant under rotations in SO(4) (see also the next Section for the physical significance of this invariance). Note that, although $\alpha_{\mathbf{M}}$ and $\alpha_{\mathbf{E}}$ in Fig. 4 are parallel, in general their magnitudes are not equal.

4 Energy and momentum

Writing the Minkowski 4-vector for energy-momentum

$$(m_0 c)^2 = (E/c)^2 - p^2 \tag{12}$$

or alternatively

$$(m_0 c)^2 = (\gamma m_0 c)^2 - (\gamma m_0 v)^2 \tag{13}$$

in Euclidean form yields:

$$(\gamma m_0 c)^2 = (m_0 c)^2 + (\gamma m_0 v)^2 \tag{14}$$

The Euclidean form becomes transparent if the identity $c = \gamma\chi$ is used:

$$(\gamma m_0 c)^2 = (\gamma m_0 \chi)^2 + (\gamma m_0 v)^2 \tag{15}$$

saying that the 4D momentum is the vector sum of spatial and proper time momentum. Equation (14) does however not yield an invariant. The factor γ, resulting from the Minkowski 4-velocity prohibits this.

The Euclidean relativistic Lagrangian for a freely moving particle in 4D,

$$\Lambda = m_0 c^2 \tag{16}$$

is a constant of motion as a result of the universal velocity magnitude c for the free particle in 4D space-time (see also the derivation of Montanus in [2]). This shows that, just as in the Euclidean 4-velocity, γ must be left out in the Euclidean form, which again yields $m_0 c$ as an invariant:

$$(m_0 c)^2 = (m_0 \chi)^2 + (m_0 v)^2 \tag{17}$$

Note that the invariance of the 4D momentum is also consistent with the invariance of the acceleration α_E that was discussed in the previous Section. Since the acceleration is always orthogonal to the velocity, the magnitude of the momentum vector will not change.

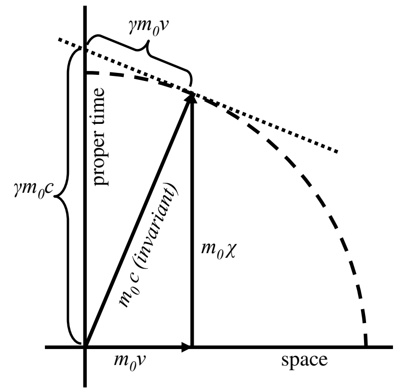

Figure 5: Minkowski and Euclidean components for energy-momentum.

Figure 5 again visualizes the geometries.

5 Current density

The relativistic density ρ of a collection of moving charges is defined as $\gamma\rho_0$, where ρ_0 is the charge density in the rest frame of the moving charges, also referred to as *proper density*. A derived quantity is the current density 4-vector J_μ that is defined as:

$$J_\mu = \rho_0 \frac{dx_\mu}{d\tau} \tag{18}$$

and is constructed quite similar to the energy-momentum 4-vector:

$$p_\mu = m_0 \frac{dx_\mu}{d\tau}. \tag{19}$$

The current density 4-vector can be rewritten into Euclidean form in the same way as with the energy-momentum 4-vector:

$$\begin{aligned}(\rho_0 c)^2 &= (\gamma\rho_0 c)^2 - (\gamma\rho_0 v)^2 \\ (\gamma\rho_0 c)^2 &= (\rho_0 c)^2 + (\gamma\rho_0 v)^2 \\ (\gamma\rho_0 c)^2 &= (\gamma\rho_0 \chi)^2 + (\gamma\rho_0 v)^2\end{aligned} \tag{20}$$

saying that the 4-dimensional current density is the vector sum of the current density in space and the current density in the proper time dimension. A similar effect of the factor γ is seen in the current density as in the energy-momentum 4-vector and this suggests that there will also be a justification that allows to leave out this factor in the equation for current density to reach an invariant result, equivalent to the relativistic Lagrangian for energy-momentum. That relativistic Lagrangian more or less represents the view on energy that a 'Hyper-spacelander' with full 4D observational skills would have. Such an upgrade from 3D to 4D observational skills eliminates length contraction effects (length is invariant in 4D) and for the 4D observer this turns the relativistic charge density ρ into the proper density ρ_0. It is therefor justified to leave out γ in Eq. (20).

With a *single* unit of charge the charge density will be the same for all inertial frames (charge is invariant). The equation for a single charge reads:

$$(\rho_0 c)^2 = (\rho_0 \chi)^2 + (\rho_0 v)^2 \tag{21}$$

and is the only equation that is on equal footing with the Euclidean 4-vector for momentum, Eq. (14), when applied to a single elementary mass particle. It strikes that the factor γ is eliminated automatically here. There is no need for an upgrade from 3D to 4D observational skills to justify any omission of γ like was done for the energy-momentum 4-vector. This markedly distinguishes the properties *charge and current* in the electromagnetic field from the properties *mass and energy-momentum* in the gravity field and seems to suggests a dimensional hierarchy, *i.e.*, the electromagnetic field seems to have one less dimension than gravity.

6 Electromagnetic potential

Finally, the potential 4-vector $A^\mu = (\phi, c\mathbf{A})$ for a uniformly moving charge could be rewritten in Euclidean form using the same method as used so far:

$$\phi^2 = K^2 + c^2 \left(A_x^2 + A_y^2 + A_z^2\right) \tag{22}$$

with K still to be determined as the temporal component in the Euclidean form. The magnitude of the vector potential $\mathbf{A}$ is

$$\sqrt{A_x^2 + A_y^2 + A_z^2} = \frac{v}{c^2}\phi \tag{23}$$

while ϕ for a moving charge can be written in terms of the *retarded* potential ϕ_r:

$$\phi = \gamma \phi_r. \tag{24}$$

Using these identities, K can be determined as:

$$K^2 = \gamma^2 \phi_r^2 \left(1 - \frac{v^2}{c^2}\right) = \phi_r^2 \tag{25}$$

The Euclidean form thus reads:

$$\phi^2 = \phi_r^2 + c^2 \left(A_x^2 + A_y^2 + A_z^2\right) \tag{26}$$

The variable ϕ is however *not* invariant. In order to make the 4-vector invariant under rotations in SO(4) (the Euclidean equivalent of Lorentz transformations, see also [3]) the expression would have to be multiplied with $1/\gamma$ but this is inconsistent with the approach in the other 4-vectors where the aim was to get rid of the gamma's in the Euclidean expressions. This raises the question whether the classical potential 4-vector could be the Euclidean form *already*, although at first sight this seems in conflict with the traditional $+ - - -$ pattern in its components, necessary to yield an invariant.

Various operations on the potential 4-vector, like e.g. the derivation of the fields of $\mathbf{E}$ and $\mathbf{B}$ in $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, involve the operator ∇_μ which is defined as:

$$\nabla_\mu = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \tag{27}$$

The derivative with respect to t in $\partial/\partial t$ is inconsistent with the derivation with respect to τ in Minkowski 4-vectors but consistent with the Euclidean 4-vectors. Furthermore, the components ϕ and $\mathbf{A}$ both already have intuitive physical meanings, showing in particular from operations like $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The pattern $+ - - -$ therefor seems an intrinsic property of electromagnetic potentials rather than being related to Minkowski geometry.

References

[1] R.F.J. van Linden, "Euclidean Special relativity", available at www.euclideanrelativity.com (Sep 2017)

[2] H. Montanus, "Proper time formulation of relativistic dynamics", Foundations of Physics **31** (9) 1357-1400 (September 2001)

[3] A. Gersten, ”Euclidean special relativity”, Foundations of Physics **33** (8) 1237-1251 (August 2003).